Designing Order-Book Transparency in Electronic Communication Networks

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Abstract

A significant fraction of trade in stock exchanges (e.g., Euronext and NASDAQ) involves "iceberg orders," which are orders to sell or buy a certain number of shares with the caveat that only a part of that number be made public. This paper provides a normative justification for the lack of transparency in this kind of orders: imperfect disclosure is shown to be a necessary feature of any optimal mechanism when the asset's potential buyers must incur a cost in order to become active and learn their valuations for the asset. This finding raises a caveat for regulation that seeks to mandate the open order-book or otherwise increase the pre-trade transparency of stock exchanges.

1 Introduction

In anonymous electronic exchanges (e.g., NASDAQ, Euronext, Instinet, Toronto Stock Exchange, London Stock Exchange, and XETRA), traders may choose to disclose their trading intentions only partially, using iceberg orders. When a trader submits an **iceberg order**, he indicates to the exchange his intended quantity, the worst price at which he is willing to trade this quantity, and a (typically) smaller quantity to be initially disclosed to other traders. He thus hides from those other traders the true size of his order. Some exchanges even restrict pre-trade disclosure outright; for instance, Pipeline (an electronic trading platform) hides most submitted orders, and indicates

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to traders only the arrival of large, appropriately priced orders, without disclosing their exact content. This paper asks two normative questions inspired by the described practices of partial information disclosure: Which features of the economic environment make it optimal for the exchange to enable traders to disclose their trading intentions only partially? When can optimal disclosure be implemented using iceberg orders and other instruments commonly provided by financial exchanges?

These questions are significant because partial information disclosure in financial exchanges is ubiquitous. Seventy-four percent of orders in 300 stocks traded on the Paris Bourse were not fully disclosed when the remaining order size exceeded 500,000 French Francs; the corresponding figure was 13% for the Toronto Stock Exchange Computer Assisted Trading System (Harris, 1996). Hidden liquidity was 22% of the inside depth of the NASDAQ 100 (Tuttle, 2003). In 2011, in the United States, about 12% of equity trading volume was executed in exchanges that did not publicly display any orders (Zhu, 2013).

Existing literature seeks to explain why traders use iceberg orders, but does not explain the institutional reasons for why such orders would be made available. Prevailing explanations for the use of iceberg orders rely on a common element in traders' valuations: (i) the threat of the "price impact" from a large trade (Anand and Weaver, 2004; Esser and Mönch, 2007), (ii) the threat of "picking-off" the option generated by a posted limit order (Aitken et al., 2001; Buti and Rindi, 2009), and (iii) the deterioration of the terms of trade due to adverse selection (Moinas, 2010). To uncover the institutional reasons for iceberg orders, the mechanism-design approach is necessary.

The paper's questions are addressed in the bilateral-trade model of Myerson and Satterthwaite (1983) extended to allow for trade in divisible quantities and to enable the buyer to undertake a costly investment.[1] The model (introduced in Section 2) has a seller and a so-called latent buyer. The seller owns a unit of the asset. The buyer demands at most a unit. The traders' valuations for the asset are their private information and are distributed independently. (Independence means that each trader is a liquidity trader, motivated by the need to hedge his idiosyncratic risk.) The

[1] McAfee (1991) and McKelvey and Page (2002) extend the model of Myerson and Satterthwaite (1983) by allowing for divisible quantities and non-linear payoffs in these quantities.

buyer is **latent** because he finds that assessing the asset's value and submitting an order is costly.[2] The seller knows his valuation initially, whereas the latent buyer learns his valuation only after he invests. This investment is also a pre-requisite for trade. The investment thus conflates the buyer's participation and information-acquisition decisions, which is a convenient modelling device.

The exchange mediates and enforces trade. In a direct revelation mechanism, the seller reports his valuation to the exchange, which recommends whether the buyer should invest. If he invests, the buyer reports his valuation. Then the exchange enforces trade, which consists of payments and the allocation of the asset.

In an indirect implementation, the exchange's trading mechanism shall be called an **electronic communication network** (ECN), a term used in practice. In this implementation, the seller submits iceberg orders to the ECN. The orders are stored in an **order book**, whose content is displayed to the buyer as described by each iceberg order. A special case of an iceberg order is a **limit order**, which is an iceberg order that specifies that the entire order be disclosed to the buyer. When all orders are limit orders, the order book is **open**. The order book is **hidden** if each submitted iceberg order specifies that nothing be disclosed to the buyer. The order book's displayed content helps the buyer decide whether to invest. If he invests, the buyer submits an order, and trade ensues. The exchange may subsidize (or tax) submitted orders.

The exchange maximizes a weighted sum of two terms: its revenue (i.e., the sum of the payments collected from traders) and the payoffs of the traders themselves. The exchange's concern for traders' payoffs can be motivated by regulatory or competitive pressure, is not required for the results, and is included only to highlight their generality. The paper's main result, Theorem 1 (Section 3), relies on three assumptions: (i) the exchange cannot directly control the buyer's investment; (ii) the exchange's objective function puts greater weight on revenue than on traders' payoffs; and (iii) the exchange cannot sell information to the buyer about the order book's content. Under these assumptions, Theorem 1 shows that any optimal order book is partially transparent.

The reasons for the theorem's conclusion are these. Because of assumption (i), the order book must disclose enough information to the buyer about the seller's orders so as to induce the in-

[2] Harris (2003, p. 323) describes latent traders thus: "Traders who would be willing to trade if asked, but who have not yet issued trading orders, have *latent trading demands*. They may not issue orders because writing orders is costly, or because they do not realize that they are willing to trade. When the probability of trading is small, traders often do not issue orders because they are costly to manage. ... Forming opinions about thousands of securities is costly. Instead, [latent traders] often wait until events force them to think about trading opportunities."

tended action (investment or no investment). The intended action varies with the seller's valuation. Hence, a hidden order book, which would disclose nothing, is suboptimal. Under assumption (ii), the exchange gains from charging the buyer more, but risks discouraging him from investing if his expected payoff becomes too low. To prevent the buyer from realizing when his payoff is particularly low, the exchange partially conceals from the buyer the seller's submitted orders. So an open order book is also suboptimal. Finally, assumption (iii) rules out the optimality of the buyer's fully-informed, ex-post efficient investment decision, which would prevail if the buyer's entire surplus could be extracted by an appropriately set fee for information.

Assumptions (i)–(iii) are natural in financial markets. Assumption (i) means the exchange cannot force a buyer to trade or acquire information if the buyer finds that doing so is unprofitable. Regarding assumption (ii), it holds, for instance, when the exchange is a revenue-maximizing monopolist, who puts no weight on traders' payoffs. For assumption (ii) to fail, the exchange must maximize the total surplus (in which case, all payments cancel out). This case is not particularly relevant from a practical point of view, however, because the exchange's implied revenue in any optimal mechanism would be negative, as Corollary 1 indicates. Finally, assumption (iii) holds if the buyer must invest just to be able to transact with the exchange. Assumption (iii) can be discarded without compromising Theorem 1's conclusion if the model is enriched in a certain way by endowing the buyer with a private signal about his valuation. The essential role of assumption (iii) is to preclude the exchange from extracting the buyer's entire surplus from investment.

For a policy implication of the main result, suppose a regulator mandates an open order book, and the exchange maximizes traders' expected payoffs subject to breaking even. The breaking-even constraint can be equivalently captured by assuming instead that the exchange maximizes a weighted sum of its revenue and traders' payoffs with the weights satisfying assumption (ii). Then, Theorem 1 implies that the imposition of full transparency reduces the exchange's payoff, the weighted sum. Because the revenue must remain at zero (by the breaking-even assumption), traders' payoffs must decrease. Thus, the imposition of full transparency benefits no one.[3]

When the exchange can subsidize (or tax) submitted orders, the solution to the mechanism-design problem can be implemented in an ECN that uses standard trading instruments: limit and iceberg orders (Theorem 2). The key feature of the solution to the mechanism-design problem that

[3]The ECN prevailing when full transparency is imposed is derived in Section B.5 of the Online Appendix.

appears in this implementation is the buyer's inability to infer the seller's valuation (and hence his full trading intention) just by looking at the order book. The seller's use of iceberg orders hinders the buyer's inference.

The conclusion that any open order book is suboptimal must be qualified. One qualification, already mentioned, is that the exchange's ability to sell information to the buyer about the seller's submitted orders would make it optimal for the exchange to induce the buyer to take the fully informed investment decision. Two more qualifications, explored in Section 4, concern competition, among exchanges or among traders.

Modelling exchange competition is subtle, and the approach this paper takes are reduced-form. Consider the model's specification in which the exchange's objective function is such that, at an optimal ECN, the exchange's expected revenue is zero. This outcome can be interpreted as arising from unmodelled exchange competition for traders, which drives the exchange's revenue down to zero. This competition leads to an ECN that does not admit an open order book, as has already been argued.

Competition may lead to an open book, however, under an additional assumption of the lack of commitment. In particular, Section 4.1 assumes that as the exchange maximizes its expected payoff, it cannot commit not to disclose the seller's orders stored in the order book. In addition, having observed the disclosed orders, the buyer can defect to an unmodelled competing exchange with a probability that is decreasing in his expected continuation payoff in the modelled exchange. Unravelling towards an open order book ensues (Theorem 3). Indeed, unable to commit, the exchange discloses the seller's orders when they are priced low enough, in order to reduce the probability of the buyer's defection. This incentive to disclose prevents the exchange from strategically pooling low-priced and high-priced orders when this pooling is ex-ante optimal. Thus, iceberg orders have no value, simply because the exchange cannot commit to enforcing them.

With trader competition, the exchange's benefit from offering iceberg orders diminishes, even without exchange competition and commitment problems. Trader competition is modelled in Section 4.2 by introducing multiple sellers and multiple buyers. As the number of traders grows, the exchange's gain from hiding the order book vanishes because each trader becomes "informationally small" (Theorem 4).

Related Literature: The closest literature related to iceberg orders has already been cited. Other relevant literature pertains to mechanism design. Existing papers that apply mechanism design to market microstructure are Biais et al. (2000) and Lu and Robert (2001). To my knowledge, mechanism design has not been applied to study iceberg orders.

The paper's main insights rely on classic results. That it may be harder to implement an intended outcome when a mechanism discloses more information is known from discussions of correlated equilibrium (Aumann, 1974), and of the Revelation Principle for games in which players take private actions (Myerson, 1982; Myerson, 1986). Even though the Revelation Principle says that there is no need to disclose more information than is necessary to induce the desired action, it does not say when disclosing more is detrimental, which depends on the application.

Calzolari and Pavan (2006a) have studied optimal disclosure. In a model of trade with resale, they find that partially disclosing the valuation of the first buyer (with whom the seller can contract) to the second buyer (with whom the seller cannot contract) may be optimal for the seller. Partial disclosure enables the seller to affect the behavior of the second buyer in the resale market, which the seller cannot control directly.[4] The action that the mechanism can affect only indirectly, through information disclosure, is resale in the model of Calzolari and Pavan (2006a), and is the buyer's investment in the present paper.

Laffont and Tirole (1993) discuss examples of value-altering investments, whereby private actions affect private valuations. In their Section 1.8, an agent makes a private investment that affects his value of the future relationship; because the principal's information about the effectiveness of investment and the value of the relationship is either the same as, or is inferior to, the agent's information, there is no role for information disclosure before investment. The same observation about disclosure applies to Crémer et al. (1998), in whose model the principal induces the agent to invest in information, which indirectly enhances the agent's value of the relationship. In the present paper, by contrast, the principal (exchange) is informed about the value of the relationship (trade), and influences the agent's (buyer's) investment by disclosing information.

Finally, the analysis of optimal mechanisms with multiple traders contributes to the literature on search auctions (by McAfee and McMillan, 1988; Crémer et al., 2003; and Crémer et al., 2007),

[4]Similarly, in the sequential common agency game of Calzolari and Pavan (2006b), the desire of the principal who moves first to affect the behavior of the principal who moves second compels the first principal to partially disclose the agent's type.

from which the present paper differs by assuming that multiple units of the asset must be allocated, that these are initially owned by disparate sellers whose valuations are their private information, and crucially, that latent buyers make private investments.[5] Without private investments, the irrelevance-of-disclosure result of Crémer et al. (2007) would have applied.

2 The Model

The model extends the bilateral-trade model of Myerson and Satterthwaite (1983) by accommodating the buyer's investment and the seller's payoff that is nonlinear in the amount of the asset.

2.1 Environment

Market Participants: An exchange mediates trades between a seller and a latent buyer. The seller owns a unit of a divisible asset. The buyer does not own any asset.

Investment: Buyer i privately takes an action $a \in \{0, 1\}$, where $a = 1$ means "invest" and costs $c > 0$, and $a = 0$ means "do not invest" and costs nothing.[6]

Private Information: Each trader privately observes his valuation, or type. The seller's type s is distributed on $\Theta_s \equiv [0, 1]$ according to a c.d.f. F_s with a positive and smooth p.d.f. f_s. If the buyer invests, his type b is distributed on $\Theta_b \equiv [0, 1]$ according to a c.d.f. F_b and a positive and smooth p.d.f. f_b with $\int b f_b(b)\, \mathrm{d}b > c$.[7] If the buyer does not invest, $b = 0$.

The buyer's investment plays two roles: (i) costly submission of an order and (ii) costly information acquisition. The assumption that $a = 0$ implies $b = 0$ captures role (i) because in any optimal mechanism, a buyer with $b = 0$ will never wish to submit an order, which is equivalent to wishing but being unable to submit. For role (ii), one can imagine that the buyer values the asset at $\int b f_b(b)\, \mathrm{d}b$ unless he invests, but is unable to trade and enjoy this valuation because of (i), so one might as well set $b = 0$ for the buyer who does not invest.

[5] Also in the model of Board and Skrzypacz (2010), multiple items are auctioned off. In their model (by contrast to most of the search literature), the search cost is multiplicative.

[6] The assumption that only the buyer invests delivers analytical tractability, and should not be interpreted literally. It is only essential that some trader invests. Assuming that the seller, who already holds the asset, may have invested in the past and need not do so again is plausible (and analytically convenient).

[7] When $\int b f_b(b)\, \mathrm{d}b < c$, the investment cost is prohibitive, meaning no mechanism can induce a buyer to invest.

All types are independent (conditional on the buyer's action). All probability distributions are regular in the sense that $u + F_s(u) / f_s(u)$ and $u - (1 - F_b(u)) / f_b(u)$ are strictly increasing in u.

Trades: A trade is a collection of the seller's and the buyer's payments (t_s, t_b) and a collection of the seller's and the buyer's nonnegative asset holdings (x_s, x_b) that are feasible: $x_s + x_b = 1$.

Traders' Payoffs: Each trader values at most one unit of the asset. Fix any trade $(t_s, t_b x_s, x_b)$. The payoff of a type-b buyer who takes action a is $bx_b - t_b - ca$. The payoff of a type-s seller is $sx_s + \psi(x_s) - t_s$, where ψ is either identically zero or is in a set Ψ that is defined to be the set of smooth, strictly increasing, and strictly concave functions. Both traders are expected-utility maximizers.

The possible nonlinearity (captured by $\psi \in \Psi$) of the seller's payoff may arise from his opportunity cost of selling the asset in another market, with a price impact. This non-linearity will turn out to be necessary for illustrating the use of iceberg orders that have both visible and hidden parts. For parsimony, buyers' payoffs are linear.

The Exchange's Payoff: The exchange's payoff is a weighted sum of its revenue and of traders' payoffs:

$$t_s + t_b + \delta_s(sx_s + \psi(x_s) - t_s) + \delta_b(bx_b - t_b - ca), \tag{1}$$

where $t_s + t_b$ is the revenue, and δ_s and δ_b in $[0, 1]$ are the weights assigned to the seller's and buyer's payoffs. (The case when $\max\{\delta_s, \delta_b\} > 1$ is uninteresting because then the exchange can increase its payoff without bound by disbursing cash.) The exchange's objective function is the expectation of (1). When $\delta_s = \delta_b = 1$, the objective function is the **total surplus**. The exchange's budget is balanced, in surplus, or in deficit if its expected revenue $\mathbb{E}[t_s + t_b]$ is, respectively, zero, positive, or negative.

2.2 Trading Mechanisms

The exchange may propose any mechanism in which the seller and the buyer engage in mediated communication, followed by the buyer's decision whether to invest, followed by further mediated communication culminating in a trade enforced by the exchange. The Revelation-Principle logic applies. Without loss of generality, when looking for a trading mechanism that maximizes its expected payoff, the exchange can restrict attention to direct mechanisms. In these mechanisms, the seller confidentially reports his type to the exchange, and the exchange recommends an action to the buyer. If the buyer invests, he reports his type to the exchange, and the exchange enforces trade. In equilibrium, having observed his type, each trader is willing to participate in the mechanism and to report truthfully, and the buyer abides by the seller's recommendation. In other words, the mechanism is **voluntary**, **truthful**, and **obedient**.[8]

The exchange designs, announces, and commits to a mechanism. A mechanism that maximizes the exchange's expected payoff is **(exchange)-optimal**. A mechanism is **efficient** if it is optimal for $\delta_s = \delta_b = 1$. A mechanism is **revenue-maximizing** if it is optimal for $\delta_s = \delta_b = 0$.

The paper is concerned with optimal mechanisms that resemble the mechanisms used in practice, especially in terms of admissible trading orders and in the way in which the information about submitted orders is disclosed. Such indirect mechanisms shall be called **electronic communication networks** (ECNs) and, whenever possible, shall use limit and iceberg orders, stored in the **order book**. An order book is **open** if it fully displays every order as soon as it has been submitted. For instance, an order book comprising exclusively limit orders is open. An order book is **hidden** if all orders are submitted confidentially and are not displayed in the order book. An order book comprising exclusively iceberg orders with no visible part is hidden. A **partially transparent** order book is neither open nor hidden. An order book containing an iceberg order with both hidden and visible parts is partially transparent. An **optimal** order book is an order book in some optimal ECN.

3 Order-Book Transparency in Optimal ECNs

This section shows that any optimal order book is partially transparent. This result is established by first solving the exchange's problem by maximizing over the set of direct mechanisms, and then showing that the value of the exchange's objective function falls if the seller's type is always revealed to the buyer before the buyer decides whether to invest.

[8]The described mechanism is sequential. Because the value to the exchange from the buyer's investment depends on the seller's type, the exchange benefits from learning the seller's type first, and only then recommending whether the buyer should invest. Restricting attention to mechanisms in which both traders report their types simultaneously would entail loss in generality. The described mechanism is deterministic; the trade and the recommendation to the buyer are deterministic functions of traders' reports. In the present environment, the optimality of deterministic trades is a standard result (see, e.g., Fudenberg and Tirole, 1991, pp. 306–307, or Laffont and Tirole, 1993, p. 120). The optimality of the deterministic recommendation to the buyer relies on the continuum of the seller's types and is established in Online Appendix B.3.

3.1 Direct Mechanisms

To formally express the constraints imposed by the requirement that a direct mechanism be voluntary, truthful, and obedient, additional notation shall be introduced. Let $x(s,b)$, $t_s(s,b)$, and $t_b(s,b)$ be, respectively, the amount of the asset transferred from the seller to the buyer, the seller's payment, and the buyer's payment when the seller reports type s and the buyer reports type b. (Thus, $x \equiv x_b = 1 - x_s$.) Define $\phi(s)$ to be the probability that the buyer is recommended to invest when the seller's type is s.

Let $U_b(b)$ denote the buyer's expected equilibrium payoff, or **rent**, from trade gross of the investment cost (if any), conditional on the buyer's type being b and conditional on the exchange recommending that the buyer invest:

$$U_b(b) \equiv \max_{\hat{b}\in\Theta_b} \frac{\int_{\Theta_s} \left(bx\left(s,\hat{b}\right) - t_b\left(s,\hat{b}\right)\right)\phi(s) f_s(s)\,\mathrm{d}s}{\int_{\Theta_s}\phi(s) f_s(s)\,\mathrm{d}s}. \tag{2}$$

In (2), the buyer uses Bayes's rule to compute the posterior probability distribution of the seller's type conditional on receiving the recommendation to invest. The buyer chooses his type report $\hat{b}$ so as to maximize his expected payoff.

The type-s seller's expected equilibrium payoff is:

$$U_s(s) \equiv \max_{\hat{s}\in\Theta_s}\left\{\phi(\hat{s})\int_{\Theta_b}\left(s\left(1-x(\hat{s},b)\right)+\psi\left(1-x(\hat{s},b)\right)-t_s(\hat{s},b)\right)f_b(b)\,\mathrm{d}b+\left(1-\phi(\hat{s})\right)\left(s+\psi(1)\right)\right\}, \tag{3}$$

where the integral is the seller's expected payoff when the buyer invests, and $s+\psi(1)$ is the seller's payoff from retaining the good when the buyer does not invest.

A direct mechanism is **voluntary** if, for all $b\in\Theta_b$ and all $s\in\Theta_s$,

$$\textbf{PC}: U_b(b)\geq 0 \qquad \text{and} \qquad U_s(s)\geq s+\psi(1).$$

The mechanism is **truthful** if

IC : b and s solve the maximization problems in (2) and (3).

The mechanism is **obedient** if the buyer abides by the recommendation to invest:

$$\textbf{OC}: \int_{\Theta_b} U_b(b) f_b(b)\,\mathrm{d}b - c \geq U_b(0).$$

3.2 The Optimal Allocation of the Asset

Following Myerson and Satterthwaite (1983), the problem of solving for an optimal mechanism can be decomposed into two steps: finding an optimal allocation and constructing the supporting payments. For the first step, the exchange's payoff-maximization problem shall be transformed into a virtual-surplus maximization problem subject to certain monotonicity constraints. The standard part in the first step is to use **IC** to substitute out the transfers from the exchange's objective function.[9] The less standard part follows Crémer et al. (1998) by combining the objective function with **OC** to form a Lagrangian. For this part, let $\lambda \int_{\Theta_s} \phi(s) f_s(s) \, \mathrm{d}s$ denote the Lagrange multiplier on **OC**. (The integral multiplying λ is a convenient normalization.) The emergent Lagrangian involves the seller's and the buyer's virtual valuations, denoted by $\eta_s(s)$ and $\eta_b(b)$, which are as in Theorem 1 of Myerson and Satterthwaite (1983) except for λ and (δ_s, δ_b):

$$\eta_s(s) \equiv s + (1 - \delta_s) \frac{F_s(s)}{f_s(s)} \qquad \text{and} \qquad \eta_b(b) \equiv b - (1 - \delta_b - \lambda) \frac{1 - F_b(b)}{f_b(b)}. \tag{4}$$

In (4), the λ that enters $\eta_b(b)$ counters a revenue-maximizing exchange's tendency to depress the buyer's payoff; this tendency may discourage the buyer's investment. A positive δ_b counters the same tendency, but now in recognition of the exchange's direct concern for the buyer's payoff. The role of δ_s in $\eta_s(s)$ is analogous.

Define $\mathcal{L}$ to be the Lagrangian (or "virtual surplus") for the exchange's expected-payoff maximization problem. The Lagrangian is evaluated at the saddle-point value of the multiplier λ:[10,11]

$$\mathcal{L} \equiv \int_{\Theta_s} \phi(s) M(s) f_s(s) \, \mathrm{d}s, \qquad \text{where} \tag{5}$$

$$M(s) \equiv \int_{\Theta_b} \left(x(s,b) \left(\eta_b(b) - \eta_s(s) \right) + \psi(1 - x(s,b)) - \psi(1) \right) f_b(b) \, \mathrm{d}b - \hat{c}, \tag{6}$$

and $\hat{c} \equiv (\delta_b + \lambda) c$. The requisite monotonicity (or global incentive-compatibility) constraints require the seller's and the buyer's expected allocations (read off (2) and (3)),

$$\phi(s) \int_{\Theta_b} (1 - x(s,b)) f_b(b) \, \mathrm{d}b + 1 - \phi(s) \qquad \text{and} \qquad \frac{\int_{\Theta_s} x(s,b) \phi(s) f_s(s) \, \mathrm{d}s}{\int_{\Theta_s} \phi(s) f_s(s) \, \mathrm{d}s}, \tag{7}$$

to be nondecreasing in traders own types s and b, respectively.

[9]See, for example, Fudenberg and Tirole (1991, pp. 284-8) or the Constraint Simplification Theorem of Milgrom (2004), which relies on the Envelope Theorem of Milgrom and Segal (2002).

[10]The necessity and sufficiency of Lagrangian analysis is Lemma 3 in the Online Appendix.

[11]The Lagrangian (5) omits the additive terms that are independent of the exchange's choices.

The term $M(s)$ in (6) can be interpreted as the exchange's expected payoff from recommending investment when the seller's type is s. The integrand in $M(s)$ is the exchange's gain from mediating a trade of size $x(s,b)$.[12] The augmented cost $\hat{c}$ reflects the exchange's direct concern (captured by $\delta_b c$) and indirect concern (captured by λc) for the buyer's payoff, the latter arising from the need to provide incentives for the buyer to invest.

If pointwise maximization of $\mathcal{L}$ is justified, the exchange optimally sets:

$$x(s,b) = \arg\max_{z\in[0,1]} \{z(\eta_b(b) - \eta_s(s)) + \psi(1-z)\} \qquad \text{and} \qquad \phi(s) = [M(s) > 0], \tag{8}$$

where the bracket notation is used for the indicator function (i.e., $[M(s) > 0]$ is 1 if $M(s) > 0$, and is 0 otherwise). Because M is decreasing in s, there exists s^* satisfying $M(s^*) = 0$ such that $\phi(s) = [s < s^*]$. Thus, the buyer invests whenever the seller's valuation is sufficiently low. It can be verified that the allocation rule implied by (8) satisfies the monotonicity of expected allocations in (7), thereby justifying the pointwise maximization of $\mathcal{L}$.

To completely characterize an optimal mechanism, it remains to specify payments supporting the allocation rule in (8). This task is postponed until Section 3.4, which concerns implementation.

3.3 Any Optimal Order Book Is Partially Transparent

When $\delta_b < 1$, the exchange's payoff is increasing in the buyer's payment. This payment is positive only if the buyer invests. Hence, the exchange may instruct the buyer to invest even when the buyer loses from investing. To invest obediently, the buyer must believe that investing pays, at least on average, over the seller's types. The exchange can persuade the buyer to invest by partially concealing from him the seller's types. Indeed, the exchange does so in any optimal mechanism (and hence in any ECN), as the following theorem shows. In the theorem's statement, an investment cost c is **nonprohibitive and nontrivial** if, for any δ_s and δ_b, in any optimal mechanism, the buyer sometimes invests and sometimes does not, that is, if $s^* \in (0,1)$.[13]

Theorem 1. *If $\delta_b < 1$ and the investment cost is nonprohibitive and nontrivial, then the order book in any optimal electronic communication network is partially transparent.*[14]

[12] The integrand in $M(s)$ is the analogue of the term inside the expectation in equation (2.9) of McKelvey and Page (2002), who consider trade in continuous quantities, but do not model costly investment.

[13] When $\psi \equiv 0$, any c satisfying the maintained assumption $0 < c < \int b f_b(b)\, db$ is both nontrivial and nonprohibitive.

[14] All omitted proofs are in the Appendix.

The suboptimality of a hidden order book is an immediate consequence of $s^* \in (0,1)$. To induce investment for some seller's types, but not others, the order book's appearance to the buyer must vary with the seller's type.

The proof of the suboptimality of an open order book proceeds by showing that, for an s slightly below s^*, any optimal mechanism induces the buyer to invest, even though given this s, the buyer loses from investing. The buyer obeys only if he does not know s, which rules out an open order book.

If, contrary to the hypothesis of Theorem 1, the exchange maximizes the total surplus (i.e., if $\delta_s = \delta_b = 1$), an optimal order book can be open.[15] Informally, each trader can be made a residual claimant to the total surplus, which the exchange maximizes. Thus, the exchange's and the buyer's incentives are aligned.

If the exchange maximizes total surplus subject to an additional constraint that is nonnegative revenue, the order book cannot be open:

Corollary 1. *Suppose the exchange maximizes total surplus subject to nonnegative revenue. If the investment cost is nonprohibitive and nontrivial, then the order book in any optimal electronic communication network is partially transparent.*

Proof. Suppose the exchange maximizes the total surplus while neglecting the constraint that the expected revenue be nonnegative. The solution is an efficient mechanism. In this mechanism, because c is non-prohibitive, the buyer invests whenever the seller's type is in $[0, s^*]$. If, an addition, $\psi \equiv 0$, then the hypotheses of Corollary 1 of Myerson and Satterthwaite (1983) hold, and the exchange's expected revenue is negative. If, instead, $\psi \in \Psi$, then Theorem 1 of McAfee (1991) applies, and the exchange's expected revenue is again negative.

Therefore, when maximizing the total surplus, the exchange cannot neglect the constraint that the revenue be nonnegative. One way to incorporate this non-negativity constraint is to reduce $\delta_s = \delta_b$ (down from 1) until the implied expected revenue equals zero. The argument in the proof of Theorem 2 of Myerson and Satterthwaite (1983) validates this approach; the argument can be straightforwardly adjusted to accommodate the buyer's investment and non-linear payoffs.

[15]This result is stated in Theorem 4, which considers multiple buyers and sellers, and has $\psi \equiv 0$.

Because $\delta_b < 1$ and the investment cost is nontrivial and nonprohibitive, the corollary's conclusion follows from Theorem 1. □

Theorem 1 and Corollary 1 both rely on the exchange's inability to make the latent buyer pay for information about the seller's order. If such a payment were feasible, the exchange would adopt an efficient ECN, whose order book can be open, and would extract the buyer's surplus by charging him a fee before he learns his type.

3.4 An Optimal ECN

By a revenue-equivalence argument, an optimal ECN is not unique. Many optimal ECNs use limit and iceberg orders. These ECNs can differ in the amount of information the order book reveals to the buyer. This section constructs a particular optimal ECN, whose order book has the minimal disclosure necessary to induce the buyer to take the intended action. This ECN uses the types of orders that are commonly used in practice: iceberg, hidden, and limit orders.

An **iceberg sell order** for quantity q at an ask p is a commitment to sell q units at price p or higher, with only $\tilde{q}$ units being initially visible to the buyer. Call q **(total) liquidity**, $\tilde{q}$ **visible liquidity**, and $q - \tilde{q}$ **hidden liquidity**. An iceberg order with $\tilde{q} = 0$ is a **hidden order**. An iceberg order with $\tilde{q} = q$ is a **limit order**.

The seller must choose a collection of iceberg orders from an admissible menu, which will be described. A collection is represented by a pair of functions $\mathbf{q} : \mathbb{R}_+ \to \mathbb{R}_+$ and $\tilde{\mathbf{q}} : \mathbb{R}_+ \to \mathbb{R}_+$, where $\mathbf{q}(p)$ is (the density of) the total liquidity placed at a price p, and $\tilde{\mathbf{q}}(p)$ is the visible liquidity at that price.[16] The buyer can submit only a limit buy order at a price p' for a unit of the asset. When the buyer's order is executed, he gets $\int_0^{p'} \mathbf{q}(p)\,\mathrm{d}p$ units and pays amount $\int_0^{p'} p\mathbf{q}(p)\,\mathrm{d}p$. A special case is when $\mathbf{q}$ has a unit mass point at some price p, in which case the buyer trades and pays p for a unit of the asset if and only if $p' \geq p$.

The following theorem exhibits one of many optimal ECNs that use limit, iceberg, and hidden orders. Common to all these ECNs is the order book's partial transparency.

Theorem 2. *An optimal electronic communication network (ECN) requires partially concealing the order book and restricting the set of the seller's and the buyer's orders. This ECN can be implemented by letting*

[16]The boldface emphasizes the distinction between functions $\mathbf{q}$ and $\tilde{\mathbf{q}}$ and scalars q and $\tilde{q}$.

the seller choose from a set of collections of iceberg orders and then letting the buyer choose any limit order. These sets of admissible orders are designed so that (i) the seller's orders, partially displayed in the order book, indicate to the buyer when to invest, (ii) an investing buyer optimally submits a limit order for one unit at the price equal to his valuation, and (iii) the equilibrium allocation is as in the optimal direct revelation mechanism. A subsidy (or tax) supplements the seller's payments prescribed by the executed orders.

The nature of the restrictions on the admissible orders alluded to in Theorem 2 depends on the seller's payoff parameter ψ. When $\psi \in \Psi$, the seller is permitted to choose any collection of iceberg orders from a menu $\{(\mathbf{q}_s, \tilde{\mathbf{q}})\}_{s<s^*}$. Each collection, indexed by s, is derived from the optimal allocation in (8) as follows:

$$\mathbf{q}_s(p) \equiv \left.\frac{\partial x(s,b)}{\partial b}\right|_{b=p} \qquad \text{and} \qquad \tilde{\mathbf{q}}(p) \equiv \inf_{s'<s^*}\{\mathbf{q}_{s'}(p)\}, \qquad s < s^*. \tag{9}$$

If $s < s^*$, the type-s seller optimally chooses collection $(\mathbf{q}_s, \tilde{\mathbf{q}})$ and collects a corresponding subsidy. In response to $\tilde{\mathbf{q}}$ displayed in the order book,[17] the buyer invests, and submits a limit order at price $p' = b$. If, by contrast, $s \geq s^*$, the seller submits no order and collects no subsidy, the buyer does not invest, and no trade occurs.

When $\psi = 0$, the seller is permitted to submit a single hidden order for a unit of the asset. If the seller's type s satisfies $s < s^*$, the seller submits a hidden order at the price $p = \eta_b^{-1}(\eta_s(s))$ and collects a corresponding subsidy. By sending an explicit alert, the ECN indicates to the buyer that he should invest.[18] The buyer invests, and bids his valuation $p' = b$. If $p' > p$, the buyer buys one unit at p. If, by contrast, $s \geq s^*$, the seller submits no order and collects no subsidy, the buyer does not invest, and no trade occurs.

Restrictions on admissible orders are common in practice. Common restrictions are the minimal tick size (or price increment) and the minimal size of the block of shares (as practiced, e.g., by FxPro and Pipeline exchanges). Restrictions are also ubiquitous in markets that admit non-traditional financial instruments, or "algorithms." Thus, Alpha Pro offers preconfigured strategies that can be customized.[19] The added fees associated with customization nudge traders towards the preconfigured strategies. Similarly, Instinet encourages traders to choose from a restricted set

[17] If $\tilde{\mathbf{q}}(p) = 0$ for all p, then no liquidity is visible, and the ECN explicitly recommends investment by sending an alert. In practice, Pipeline ECN uses such alerts.

[18] Because the seller's order is hidden, the order book has no visible liquidity whether or not the seller submits an order.

[19] See http://www.pipelinetrading.com/ for details.

of "event-driven algorithmic trading strategies with advanced liquidity-seeking and anti-gaming techniques operating within a risk-controlled framework."[20]

According to Theorem 2, the seller's payments prescribed by executed orders are supplemented by subsidies (or taxes).[21] These subsidies play two roles. The first role is to align the seller's and the exchange's objectives when the exchange's objective function differs from the seller's payoff (i.e., when $\delta_s < 1$ or $\delta_b > 0$). The second role is to align the incentives of different types of the seller, which may be necessary even if the seller's and the exchange's objectives are aligned (i.e., when $\delta_s = 1$ and $\delta_b = 0$). Both roles of subsidies are standard in mechanism design. Articulating the second role in the context of ECNs is instructive, however, as doing so helps understand what motivates different types of the seller to submit iceberg orders with the same visible part.

For simplicity, assume that $\psi \equiv 0$ and that the exchange's objective is to maximize the seller's expected payoff (i.e., $\delta_s = 1$ and $\delta_b = 0$). Let Alice be a seller whose type is close to 0. Let Bob be a seller whose type is less than, but close to, s^*. Theorem 1 requires Alice to submit a hidden order at a price that, if observed by the buyer, would have persuaded him to invest. The theorem requires Bob to submit a hidden order at a price that, if observed by the buyer, would have dissuaded him from investing. Alice's and Bob's prices are not observed, however, and the buyer invests even when the seller is Bob.

If not for sellers such as Alice, Bob would either not have been able to trade because the buyer would not have invested, or would have had to lower his price to motivate the buyer to invest. Thus, "pooling" his order with Alice's benefits Bob. Alice, by contrast, might have been better off without Bob, raising and then disclosing her price to the buyer (who would still invest), instead of lowering her price and helping sellers such as Bob by pooling with them. To induce Alice to lower her price and pool, an optimal ECN subsidizes the submission of low-priced hidden orders.

When $\psi \in \Psi$, Alice (a low-type seller) optimally submits a collection of iceberg orders at low prices and for large quantities. In this case, her subsidy's counterpart in practice is a rebate to a

[20] See http://www.instinet.com/includes/index.jsp?thePage=/html/at_index.txt.

[21] When c is small, subsidies may be redundant.

large liquidity provider.[22] Bob (a seller whose type is near s^*), in turn, pays a tax, which can be interpreted as an order-submission fee.

4 Competition and the Order-Book Transparency

One can imagine at least three types of competition in financial markets: (i) exchanges compete for traders *ex ante*, before traders have observed their types; (ii) due to limited commitment, exchanges compete also in the *interim*, after some traders have observed their types and submitted their orders; and (iii) within an exchange, traders compete with traders on the same side of the market for traders on the opposite side of the market.

The model described in Section 2 can be interpreted as arising from type-(i) competition. First, each exchange announces and commits to an ECN. Then, each trader chooses an ECN to join. Finally, trading occurs. The focus is on the equilibrium in which all traders join the ECN that promises the highest expected payoff. Competition drives each exchange's expected revenue down to zero, leading to the situation described in Corollary 1, and hence to the suboptimality of an open order book.

The remainder of this section focuses on the competitions of types (ii) and (iii), which will essentially lead to an open order book. For concreteness, it is assumed that compelled by competition, the exchange maximizes the total surplus subject to the zero-revenue condition.

4.1 An Open Order Book Prevails when Competing Exchanges Lack Commitment

Modelling type-(ii) competition is delicate and will be done in reduced form. Assume a minimal form of **limited commitment**: the exchange commits to the allocation and payment rules, but not to the order book's disclosure policy. In particular, in the *interim*—after having received the seller's order, but before having indicated to the buyer whether to invest—the exchange will disclose fully, or "flash," the seller's order if doing so improves his continuation payoff. Limited commitment alone does not undermine the (*ex-ante*) optimal ECN, because flashing can only dissuade the buyer from investing, thereby reducing the exchange's payoff.

[22]For instance, in 2009, the New York Stock Exchange introduced rebates to so-called supplementary liquidity providers, who are essentially high-frequency traders.

Interim competition is modelled in reduced form: the exchange believes that, with some probability $\varepsilon > 0$, it faces competition that offers the buyer an expected payoff that is distributed on $[0, 1]$ with a strictly increasing c.d.f. H. Hence, if the exchange promises the buyer an expected payoff $u \geq c$, the buyer invests with probability $1 - \varepsilon (1 - H(u))$. If the exchange promises an expected payoff $u < c$, the buyer does not invest.

Limited commitment and interim competition imply that the exchange can do no better than to propose an ECN with an open order book:

Theorem 3. *With limited commitment and interim competition, the exchange proposes an electronic communication network that has an open order book.*

Proof. Suppose, by contradiction, that the exchange gains strictly from pooling the orders submitted by seller types s and s'. The exchange's strict gain implies that the buyer invests with higher probability if he knows s' than if he knows s (otherwise, relabel s and s'). In that case, the exchange chooses to flash s'. An open order book prevails. □

4.2 An Open Order Book Is Approximately Optimal When Traders Are Many

For type-(iii) competition, consider the (full-commitment) model described in Section 2 with $\psi = 0$, and extend it in the obvious way so as to accommodate multiple, ex-ante identical sellers and latent buyers, all with independently distributed types. In an optimal ECN (described in Theorem 7 of the Online Appendix), all sellers submit their orders, and then buyers invest and submit their orders, sequentially, in groups, as they are alerted to trade opportunities.[23] As the number of traders grows, an open order book becomes approximately optimal:

Theorem 4. *Suppose that in an environment with S_0 sellers and B_0 latent buyers, in a surplus-maximizing electronic communication network (ECN), at least some buyer invests for some realizations of sellers' types.*[24] *Consider a sequence of replica environments with $S = \tau S_0$ sellers and $B = \tau B_0$ buyers, where $\tau = 1, 2, ...$ Then, a surplus-maximizing ECN with an open order book exists for each τ, and as $\tau \to \infty$, the exchange's expected revenue approaches zero (from below).*

[23]The assumption that an ECN can sequentially and individually alert buyers is consistent with the technology used by Pipeline Trading, which provides personalized alerts.

[24]This hypothesis rules out trivial cases.

Proof. See Online Appendix B.4. □

The intuition for the theorem's conclusion can be seen informally in a related ECN that is *approximately* efficient, *exactly* budget balanced, and alerts all relevant buyers simultaneously. This ECN publicly posts a price at which traders are invited to transact.

Assume that the Law of Large numbers applies;[25] as $\tau \to \infty$, the fractions of sellers and investing buyers willing to transact at a price p are, respectively, $F_s(p)$ and $1 - F_b(p)$. Let B' (with $B' \leq B$) denote the number of investing buyers. Feasibility requires the numbers of sellers and buyers willing to trade at price p to be equal:

$$SF_s(p) = B'(1 - F_b(p)), \tag{10}$$

where the equality is "asymptotic." The surplus-maximizing exchange chooses B' to maximize:

$$B' \int_p^1 b f_b(b)\, \mathrm{d}b - S \int_0^p s f_s(s)\, \mathrm{d}s - cB', \tag{11}$$

where p adjusts so that (10) holds. Neglecting the integer constraint and differentiating (11) with respect to B' and subject to (10) gives the exchange's gain from asking an additional buyer to invest: $\mathbb{E}_b[\max\{0, b - p\}] - c$. This gain coincides with a buyer's expected gain from investment. Thus, the exchange's and the buyers' incentives to invest are aligned even when the buyers know p; an order book can be open. Because the exchange knows the optimal B' in advance, the order book can be revealed to all B' buyers simultaneously. That is, the exchange's loss from replacing sequential alerts (as used by Pipeline) with simultaneous ones (as used in traditional order-book-driven exchanges) vanishes as the number of traders grows.

5 Conclusions

The model's main policy implication is that the partial transparency of the order book is an indispensable feature of any optimally designed exchange. The result does not depend on whether the exchange maximizes its revenue, the total surplus subject to breaking even, or something intermediate. In particular, if the exchange maximizes the total surplus subject to breaking even, then the policy of imposing full transparency would only reduce traders' payoffs. Partial transparency is

[25]The justification of the Law of Large Numbers appears in Section 5 of Gresik and Satterthwaite (1989). Lu and Robert (2001) use a related argument in their asymptotic analysis.

not an artifact of restricting attention to traditional trading instruments, as the mechanism design analysis shows, nor does it require asymmetric information about the asset's common value.[26]

The implementation of an optimal direct mechanism using traditional order types (i.e., limit and iceberg orders) suggests that iceberg orders may be imperfect safeguards of hidden liquidity if admissible orders are unrestricted. Indeed, consider a latent buyer who has not invested yet, and who, contrary to the model's assumption, can submit multiple orders, sequentially.[27] Suppose he submits a limit buy order for an amount just exceeding visible liquidity. By observing whether his order has been executed fully, he learns about the amount of hidden liquidity. With this knowledge, he can judge better whether to invest.

To make the described probing by the buyer unprofitable, some exchanges limit the minimal size of an order. [28] Probing can be also made less profitable by increasing the "tick size," the minimal price increment. Then, to gain priority in the execution of his order, a trader must improve upon the price of the best outstanding order by the amount of the tick size. This requirement imposes a cost of submitting an additional order and may discourage probing.

Even though the model justifies the use of iceberg orders, it shows they are optimal typically only when combined with subsidies and restrictions on admissible orders. Furthermore, for some payoff specifications (i.e., when $\psi \equiv 0$), iceberg orders may be unable to alert latent buyers to attractive trading opportunities, and must be supplemented by explicit alerts, such as those used by Pipeline. In this case, non-traditional trading instruments (e.g., algorithms) and non-traditional communication channels (e.g., explicit alerts) become normatively compelling.

A Appendix

A.1 Any Optimal Order Book Is Partially Transparent

Proof of Theorem 1 In an incentive-compatible and voluntary mechanism, conditional on the seller's type s, the buyer's expected rent is obtained by a standard Envelope argument (see, e.g.,

[26]The omission of asymmetric information about the common value is not meant to deny this asymmetry in practice. Instead, the model's goal is to illustrate the logic behind a partially transparent order book in the simplest possible way.

[27]Consistent with the model's assumption that the buyer who has not invested does not value the asset, assume that whatever the buyer has bought before having invested has no value to him, even if he subsequently invests.

[28]For example, Pipeline sets this size to $10,000$ shares (Ready, 2010).

Milgrom and Segal, 2002):

$$R(s) \equiv \int_0^1 x(s,b)(1-F_b(b))\,\mathrm{d}b - c. \tag{A.1}$$

When the order book is open, the buyer invests if and only if $R(s) \geq 0$. This inequality will be shown to be violated for some s for which **OC** holds and the buyer is optimally induced to invest, thus establishing the suboptimality of an open order book.

The optimal allocation rule that enters (A.1) is obtained from (8) by splitting it into two cases, depending on ψ:

$$x(s,b) = [\psi = 0]\left[b > \eta_b^{-1}(\eta_s(s))\right] + [\psi \in \Psi] \arg\max_{z\in[0,1]} \{z(\eta_b(b) - \eta_s(s)) + \psi(1-z)\},$$

where the inverse function η_b^{-1} exists because η_b is increasing (by the regularity of F_s and because $\lambda \leq 1 - \delta_b$, which follows by inspection of the Lagrangian in (5)).

The theorem will be proved with the help of two lemmas. Lemma 1 decomposes $M(s)$ (defined in (6)) into $R(s)$ and $T(s)$ defined by

$$T(s) \equiv [\psi = 0]\frac{\left(1 - F_b\left(\eta_b^{-1}(\eta_s(s))\right)\right)^2}{f_b\left(\eta_b^{-1}(\eta_s(s))\right)} + [\psi \in \Psi]\int_0^1 \frac{\partial x(s,b)}{\partial b}\frac{(1-F_b(b))^2}{f_b(b)}\mathrm{d}b.$$

Lemma 2 shows that, in this decomposition, $T(s)$ has a positive weight.

Lemma 1.

$$M(s) = (\delta_b + \lambda) R(s) + (1 - \delta_b - \lambda) T(s). \tag{A.2}$$

Proof. When $\psi \in \Psi$, $x(s,b)$ is an integral of $\partial x(s,b)/\partial b$, and integration by parts is justified,

$$\int_0^1 x(s,b) f_b(b)\,\mathrm{d}b = \int_0^1 \frac{\partial x(s,b)}{\partial b}(1-F_b(b))\,\mathrm{d}b \tag{A.3}$$

$$\int_0^1 x(s,b)\, b f_b(b)\,\mathrm{d}b = \int_0^1 \left(x(s,b) + \frac{\partial x(s,b)}{\partial b} b\right)(1-F_b(b))\,\mathrm{d}b \tag{A.4}$$

$$\int_0^1 \psi(1 - x(s,b)) f_b(b)\,\mathrm{d}b = \psi(1 - x(s,1)) + \int_0^1 \frac{\partial x(s,b)}{\partial b}\psi'(1 - x(s,b)) F_b(b)\,\mathrm{d}b, \tag{A.5}$$

where $x(s,0) = 0$ has been used. Whenever $\partial x(s,b)/\partial b \neq 0$, $x(s,b)$ satisfies the first-order condition for optimality:

$$\psi'(1 - x(s,b)) = \eta_b(b) - \eta_s(s). \tag{A.6}$$

Substituting (A.3), (A.4), (A.5), and (A.6) into (6) and rearranging gives:

$$M(s) = (\delta_b + \lambda) R(s) + (1 - \lambda - \delta_b) \int_0^1 \frac{\partial x(s,b)}{\partial b} \frac{(1 - F_b(b))^2}{f_b(b)} \mathrm{d}b$$
$$+ \int_0^1 \frac{\partial x(s,b)}{\partial b} \psi'(1 - x(s,b)) \mathrm{d}b - (\psi(1) - \psi(1 - x(s,1))),$$

where the second line is zero by the Fundamental Theorem of Calculus. Hence (A.2).

When $\psi \equiv 0$, the derivation of (A.2) is simpler and is omitted. □

Lemma 2. *If $\delta_b < 1$, then $\delta_b + \lambda < 1$.*

Proof. By inspection of the Lagrangian (5), there exists $\delta_b^* \in [0,1]$ such that for each δ_b, $\lambda = \max\{0, \delta_b^* - \delta_b\}$. It will be shown that $\delta_b^* < 1$. By contradiction, suppose $\delta_b^* = 1$, in which case $\lambda + \delta_b = 1$. Then, by Lemma 1, $M(s) = R(s)$.

Furthermore, for all $s < s^*$, $M(s) > 0$. Indeed, note that

$$M(s) \geq \mathbb{E}_{|s}[x(s^*,b)(\eta_b(b) - \eta_s(s)) + \psi(1 - x(s^*,b)) - \psi(1)] - c > M(s^*) = 0,$$

where the last inequality is strict because $x(s^*,b) > 0$ for some b. Indeed, if $x(s^*,b) = 0$ for all b, then $M(s^*) = -c < 0$, contradicting $M(s^*) = 0$.

Because $M(s) > 0$ for $s < s^*$, and because $s^* \in (0,1)$ by the hypothesis of Theorem 1, conditional on the recommendation to invest, the buyer's expected payoff is $\mathbb{E}[R(s) \mid s < s^*] = \mathbb{E}[M(s) \mid s < s^*] > 0$. The obtained inequality implies that **OC** is slack, so $\lambda = 0$ and, hence, $\lambda + \delta_b < 1$ (by $\delta_b < 1$), which disproves the contradiction hypothesis. □

One can now prove the theorem. Suppose $\psi \equiv 0$. If $\eta_b^{-1}(\eta_s(s^*)) = 1$, then $x(s^*,b) = 0$ for all b and $R(s^*) = -c < 0$. If $\eta_b^{-1}(\eta_s(s^*)) < 1$, then $F_b\left(\eta_b^{-1}(\eta_s(s^*))\right) < 1$ and $T(s^*) > 0$.

Suppose $\psi \in \Psi$. If $x(s^*,b) = 0$ for all b, then $R(s^*) = -c < 0$. Otherwise, on some nonempty interval in Θ_b, $\partial x(s^*,b)/\partial b > 0$, in which case $T(s^*) > 0$.

It will be shown that $T(s^*) > 0$ implies $R(s^*) < 0$. By Lemma 2, $\delta_b + \lambda < 1$. It is impossible that $\delta_b + \lambda = 0$; for otherwise, from $M(s^*) = 0$ and (A.2), $T(s^*) = 0$, contradicting the premise $T(s^*) > 0$. Therefore, $\delta_b + \lambda \in (0,1)$, and $R(s^*) < 0$ is implied by $T(s^*) > 0$, $M(s^*) = 0$, and (A.2) in Lemma 1.

Because $R(s^*) < 0$, one can find $\varepsilon > 0$ such that $R(s) < 0$ for all $s \in (s^* - \varepsilon, s^*)$, contradicting the optimality of an open order book. That is, any optimal mechanism recommends investment for every $s < s^*$, whereas the buyer loses from investment for any $s \in (s^* - \varepsilon, s^*)$ and would refuse to invest if s were revealed.

A.2 An Optimal ECN

Proof of Theorem 2 To complete the proof, it remains to compute the subsidies that motivate the seller to submit the orders that lead to the optimal allocation rule described in Section 3.2. These subsidies are the difference between the payments that implement the optimal allocation rule (these payments are computed using standard envelope arguments, described, e.g., by Milgrom and Segal, 2002) and the payments prescribed by the submitted iceberg orders.

By the Envelope Theorem, conditional on the recommendation to invest, a type-b buyer's expected payoff is

$$\mathbb{E}_{s|s<s^*,b}\left[\int_0^b x\left(s,p\right)\mathrm{d}p\right].$$

This expected payoff can be verified to prevail if the buyer's payment is:

$$t_b\left(s,b\right)=\int_0^b \mathbf{q}_s\left(p\right)p\mathrm{d}p, \tag{A.7}$$

where $\mathbf{q}_s$ is defined in (9) if $\psi\in\Psi$, and, by convention, $\int_0^b \mathbf{q}_s\left(p\right)p\mathrm{d}p=\eta_b^{-1}\left(\eta_s\left(s\right)\right)\left[b>\eta_b^{-1}\left(\eta_s\left(s\right)\right)\right]$ if $\psi\equiv 0$.[29] When $\psi\equiv 0$, $\mathbf{q}_s$ has a unit mass point at the price $\eta_b^{-1}\left(\eta_s\left(s\right)\right)$ (at which a type-s seller submits his hidden order) and is zero everywhere else. Because the payment in (A.7) implements the optimal allocation rule, no additional subsidy is required in order to induce a type-b buyer to submit a limit buy order at price b.

By contrast, for a type-s seller to be willing to submit $\mathbf{q}_s$, a subsidy (or tax) will be required. This subsidy is the difference between the payment collected from the buyer and the payment prescribed to the seller to maintain incentive compatibility. The expected payment that the exchange collects from the buyer when the seller's type is s ($s<s^*$) is a standard expression (see, e.g., the derivation of equation (1) in Theorem 1 of Myerson and Satterthwaite (1983)):

$$\mathbb{E}_{|s}\left[x\left(s,b\right)\left(b-\frac{1-F_b\left(b\right)}{f_b\left(b\right)}\right)\right]. \tag{A.8}$$

To compute the seller's expected payment, apply the Envelope Theorem to obtain the type-s ($s<s^*$) seller's expected equilibrium payoff:

$$\mathbb{E}_{|s}\left[\psi\left(1\right)+s+\int_s^{s^*}x\left(\tilde{s},b\right)\mathrm{d}\tilde{s}\right].$$

This expected payoff can be verified to prevail if the exchange's expected payment to the seller is:

$$\mathbb{E}_{|s}\left[-t_s\left(s,b\right)\right]=\mathbb{E}_{|s}\left[\psi\left(1\right)-\psi\left(1-x\left(s,b\right)\right)+sx\left(s,b\right)+\int_s^{s^*}x\left(\tilde{s},b\right)\mathrm{d}\tilde{s}\right]. \tag{A.9}$$

[29]When $\psi\in\Psi$ and $x\left(s,\cdot\right)$ is absolutely continuous, one can integrate (A.7) by parts to obtain $bx\left(s,b\right)-t_b\left(s,b\right)=\int_0^b x\left(s,p\right)\mathrm{d}p$, which verifies the claim about the buyer's expected payoff. When $\psi\equiv 0$, direct substitution leads to the standard expression for the buyer's payoff in an incentive-compatible mechanism, $bx\left(s,b\right)-t_b\left(s,b\right)=\max\left\{0,b-\eta_b^{-1}\left(\eta_s\left(s\right)\right)\right\}$, which again verifies the claim.

Therefore, the (possibly negative) subsidy that the exchange pays to the seller is the difference between (A.9) and (A.8):

$$S(s) = \mathbb{E}_{|s}\left[\psi(1) - \psi(1 - x(s,b)) + \int_s^{s^*} x(\tilde{s},b)\,\mathrm{d}\tilde{s} - x(s,b)\left(b - \frac{1 - F_b(b)}{f_b(b)} - s\right)\right].$$

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B Online Appendix

[BEFORE SUBMITTING, RESTORE REFERENCES (EXCLUDED IN ORDER NOT TO CONTAMINATE THE PART OF THE PAPER INTENDED FOR PUBLICATION.)]

B.1 Saddle Points and Duality

The following theorems, adapted from XXXLuenberger1969, are used to justify the Lagrangian analysis of the exchange's problem. Theorem 5 adapts Corollary 1 and Theorem 2 of XXXLuenberger1969pp.219,221 to a nonconvex maximization problem with a convex region below the graph of the primal functional. The functional is defined in the theorem's statement. The theorem gives the conditions that suffice for the solutions to a (possibly nonconvex) maximization problem to be characterized by a saddle-point condition.

Theorem 5. *Let Ω be a convex subset of a linear vector space X. Let P be the positive cone in a normed vector space Z, whose dual is Z^* and whose null vector is ν. Let the cone P be closed and contain an interior point. Let $f : \Omega \to \mathbb{R}$ be a functional and $G : \Omega \to Z$ be a mapping satisfying the regularity condition that $G(x_1) > \nu$ for some $x_1 \in \Omega$. Let $\Gamma = \{z \in Z \mid \exists x \in \Omega \text{ s.t. } G(x) \geq z\}$ be convex. Let the primal functional $\omega : \Gamma \to \mathbb{R}$, defined as*

$$\omega(z) = \sup\{f(x) \mid x \in \Omega, G(x) \geq z\},$$

be concave. Define the Lagrangian functional $\mathcal{L} : \Omega \times Z^* \to \mathbb{R}$ as

$$\mathcal{L}(x, z^*) = f(x) + \langle G(x), z^* \rangle,$$

where $\langle G(x), z^* \rangle$ denotes the value of the functional z^* at the point $G(x)$.

Then,

$$x_0 \in \arg\max\{f(x) \mid x \in \Omega, G(x) \geq \nu\}$$

if and only if there exists $z_0^* \in Z^*$ with $z_0^* \geq \nu$ such that for all $x \in \Omega$ and all $z^* \in Z^*$ with $z^* \geq \nu$,

$$\mathcal{L}(x, z_0^*) \leq \mathcal{L}(x_0, z_0^*) \leq \mathcal{L}(x_0, z^*).$$

Proof. Omitted. □

Theorem 6 establishes strict duality by adapting Theorem 1 of XXXLuenberger1969p224 to a nonconvex maximization problem with a convex region below the graph of the primal functional. This theorem establishes strict duality.

Theorem 6. *Suppose the assumptions of Theorem 5 hold and $\omega(\nu)$ is finite. Then,*

$$\omega(\nu) = \min_{\substack{z^* \in Z \\ z^* \geq \nu}} \sup_{x \in \Omega} \{f(x) + \langle G(x), z^* \rangle\},$$

and the minimum on the right is achieved by some $z_0^* \geq \nu$. Moreover, if the supremum in $\omega(\nu)$ is achieved by some $x_0 \in \Omega$, then

$$x_0 \in \arg\max_{x \in \Omega} \{f(x) + \langle G(x), z_0^* \rangle\} \qquad \textit{and} \qquad \langle G(x_0), z_0^* \rangle = 0.$$

Proof. Omitted. □

B.2 Necessity and Sufficiency of the Lagrangian

First, Lagrangian (5) shall be constructed, and then Lemma 3 shall establish that the Lagrangian can be used to characterize the solution to the exchange's problem.

By **IC** and the Envelope Theorem applied to (2) and (3),

$$U_b(b) = U_b(0) + \int_0^b \mathbb{E}_{|b',\mathcal{I}}\left[x\left(s, b'\right)\right] \mathrm{d}b' \tag{B.1}$$

$$U_s(s) = U_s(1) - \int_s^1 \left(1 - \mathbb{E}_{|s'}\left[x\left(s', b\right)\right]\right) \mathrm{d}s', \tag{B.2}$$

which imply that $U_b(b)$ is weakly increasing in b, and that $U_s(s) - s$ is weakly decreasing in s. Therefore, among all the inequalities in **PC**, it suffices to verify the inequalities for $b = 0$ and $s = 1$:

$$\mathbf{PC}' : U_b(0) \geq 0 \qquad \text{and} \qquad U_s(1) \geq 1 + \psi(1).$$

Integrating (B.1) and (B.2) by parts gives:

$$\mathbb{E}\left[U_b(b)\right] = U_b(0) + \mathbb{E}_{|\mathcal{I}}\left[x \frac{1 - F_b(b)}{f_b(b)}\right] \tag{B.3}$$

$$\mathbb{E}\left[U_s(s)\right] = U_s(1) - \mathbb{E}\left[(1 - x) \frac{F_s(s)}{f_s(s)}\right], \tag{B.4}$$

where the arguments of x have been omitted. By (B.3) and Bayes's rule, **OC** can be expressed as[30]

$$\mathbf{OC}' : \mathbb{E}\left[\phi(s)\left(x \frac{1 - F_b(b)}{f_b(b)} - c\right)\right] \geq 0.$$

Using the expressions for expected payoffs from (B.3) and (B.4) and substituting them into the exchange's payoff function (1) gives the following expression for the exchange's objective func-

[30]**OC**′ relies on the Revelation Principle for environments with private actions, which implies that in order to induce the buyer to invest, it suffices to recommend he invest, without revealing any additional information. Hence, when looking for an optimal mechanism, it suffices to construct only one **OC**′ (or **OC**).

tion:

$$\mathbb{E}\left[\begin{array}{c}\phi(s)\left(\begin{array}{c}x\left(b-(1-\delta_b)\dfrac{1-F_b(b)}{f_b(b)}-\left(s+(1-\delta_s)\dfrac{F_s(s)}{f_s(s)}\right)\right)\\+\psi(1-x)-\psi(1)-\delta_b c\end{array}\right)\\+s+(1-\delta_s)\dfrac{F_s(s)}{f_s(s)}+\psi(1)-(1-\delta_s)U_s(1)-(1-\delta_b)U_b(0)\end{array}\right],$$

which uses the fact that trade occurs only if the buyer invests.

Because $\mathbf{OC}'$ is independent of $U_b(0)$, and the exchange's objective function is weakly decreasing in $U_b(0)$ and $U_s(1)$, it is optimal for the exchange to set $U_s(1)$ and $U_b(0)$ to the lowest values consistent with $\mathbf{PC}'$. Then, after some simplification, the objective function becomes:

$$\mathbb{E}\left[\phi(s)\left(x\left(b-(1-\delta_b)\frac{1-F_b(b)}{f_b(b)}-s-(1-\delta_s)\frac{F_s(s)}{f_s(s)}\right)+\psi(1-x)-\psi(1)-\delta_b c\right)\right] \qquad \text{(B.5)}$$
$$+\,\delta_s(\mathbb{E}[s]+\psi(1)),$$

which, combined with $\mathbf{OC}'$ (with the Lagrange multiplier λ), gives the Lagrangian

$$\mathbb{E}[\phi(s)(x(\eta_b(b)-\eta_s(s))+\psi(1-x)-\psi(1)-\hat{c})]+\delta_s(\mathbb{E}[s]+\psi(1)),$$

from which $\mathcal{L}$ in (5) differs only in that it omits $\delta_s(\mathbb{E}[s]+\psi(1))$, the term that is independent of the choice of a mechanism.

In the following lemma, the exchange's **relaxed problem** consists of maximizing (B.5) subject to $\mathbf{OC}'$ while neglecting the monotonicity constraints (which require each trader's expected allocation to be weakly increasing in his type).

Lemma 3. *The Lagrangian $\mathcal{L}$ in (5) has the following properties:*

(i) (ϕ^*, x^*) is a solution to the exchange's relaxed problem if and only if there exists a Lagrange multiplier $\lambda^* \geq 0$ such that (ϕ^*, x^*) and λ^* form a saddle point of $\mathcal{L}$;

(ii) the solutions of the exchange's relaxed problem are the solutions of the exchange's (non-relaxed) problem.

Proof. Part (i) is proved by verifying that the conditions of Theorem 5 hold. Let $Z = E^1$, the one-dimensional Euclidian space; let $P = \mathbb{R}_+$ be the positive cone; let X_1 be the L_2 space on $[0,1]\times[0,1]$ with the Lebesgue measure; let Ω_1 be the subset of X_1 containing functions with the range in $[0,1]$; let X_2 be $L_2[0,1]$; let Ω_2 be the subset of X_2 containing functions with the range in $[0,1]$; define $X = X_1 \times X_2$ and $\Omega = \Omega_1 \times \Omega_2$. One can ascertain that P is closed and has an interior point, and that Ω is convex. The dual of Z is $Z^* = E^1$, and $\nu = 0$. It can be verified that the constraint functional, defined by

$$G(x,\phi) \equiv \mathbb{E}\left[\phi(s)\left(x(s,b)\frac{1-F_b(b)}{f_b(b)}-c\right)\right],$$

satisfies the regularity condition because $\int b f_b(b)\,\mathrm{d}b > c$, which is a maintained assumption.

It remains to construct the primal functional and verify its concavity. To conserve space, denote

$$n_b(b) \equiv b - (1-\delta_b)\frac{1-F_b(b)}{f_b(b)}, \qquad g(s,b) \equiv n_b(b) - \eta_s(s)$$
$$h(b) \equiv \frac{1-F_b(b)}{f_b(b)}, \qquad k \equiv \psi(1) + \delta_b c,$$

and suppress the arguments of ϕ, x, g, and h when no confusion is likely to arise. Define

$$\Gamma = \{z \in Z \mid (\exists (x,\phi) \in \Omega)\,(\mathbb{E}[\phi(xh - c)] \geq z)\},$$

which is convex, by inspection. Define the primal functional $\omega : \Gamma \to \mathbb{R}$ as

$$\omega(z) = \sup\{\mathbb{E}[\phi(xg + \psi(1-x) - k)] \mid (x,\phi) \in \Omega, \mathbb{E}[\phi(xh-c)] \geq z\}.$$

It remains to establish that ω is concave. To do so, let $\alpha \in (0,1)$. For any $((x_1,\phi_1),(x_2,\phi_2)) \in \Omega^2$, define $(\tilde{x},\tilde{\phi}) \in \Omega$ by letting, for each $(s,b) \in [0,1]\times[0,1]$,

$$\tilde{x}(s,b) = \frac{\alpha\phi_1(s)\,x_1(s,b) + (1-\alpha)\,\phi_2(s)\,x_2(s,b)}{\alpha\phi_1(s) + (1-\alpha)\,\phi_2(s)}$$
$$\tilde{\phi}(s) = \alpha\phi_1(s) + (1-\alpha)\,\phi_2(s).$$

Denote

$$f(x,\phi) \equiv \mathbb{E}[\phi(xg + \psi(1-x) - k)].$$

For any $(z_1,z_2) \in \Gamma^2$,

$$\begin{aligned}
\omega(\alpha z_1 + (1-\alpha)z_2) &= \sup\{f(x,\phi) \mid (x,\phi)\in\Omega, G(x,\phi) \geq \alpha z_1 + (1-\alpha)z_2\} \\
&\geq \sup\{f(\tilde{x},\tilde{\phi}) \mid (x_1,\phi_2),(x_2,\phi_2)\in\Omega, G(x_1,\phi_1)\geq z_1, G(x_2,\phi_2)\geq z_2\} \\
&= \sup\left\{\begin{array}{c} \alpha f(x_1,\phi_1) + (1-\alpha)f(x_2,\phi_2) \\ +\mathbb{E}\left[\tilde{\phi}\left(\psi(1-\tilde{x}) - \frac{\alpha\phi_1}{\tilde{\phi}}\psi(1-x_1) - \frac{(1-\alpha)\phi_2}{\tilde{\phi}}\psi(1-x_2)\right)\right] \mid \\ (x_1,\phi_2),(x_2,\phi_2)\in\Omega, G(x_1,\phi_1)\geq z_1, G(x_2,\phi_2)\geq z_2 \end{array}\right\} \\
&\geq \sup\left\{\begin{array}{c} \alpha f(x_1,\phi_1) + (1-\alpha)f(x_2,\phi_2) \mid \\ (x_1,\phi_2),(x_2,\phi_2)\in\Omega, G(x_1,\phi_1)\geq z_1, G(x_2,\phi_2)\geq z_2 \end{array}\right\} \\
&= \alpha\omega(z_1) + (1-\alpha)\,\omega(z_2),
\end{aligned}$$

where the last inequality follows by the concavity of ψ. Hence, ω is concave, as claimed, and part (i) follows by Theorem 5.

To show part (ii), note that for some $\delta^* \in [0,1]$, $\lambda + \delta_b = \max\left\{\delta_b^*, \delta_b\right\}$, by inspection of $\mathcal{L}$. Then, $\lambda + \delta_b \leq 1$ and the regularity of F_s and F_b imply that[31] $\eta_b(b,\lambda) - \eta_s(s)$ is strictly decreasing in s and strictly increasing in b, implying

$$x(s,b) \equiv \arg\max_{z\in[0,1]} \{z(\eta_b(b,\lambda) - \eta_s(s)) + \psi(1-z)\}$$

is weakly decreasing in s and weakly increasing in b, thereby satisfying the requisite monotonicity conditions. Then, the result follows by part (i). □

B.3 The Optimality of Deterministic Recommendations

The following lemma shows that, given the seller's type, the exchange finds it optimal to give a deterministic recommendation to the buyer about whether to invest.

Lemma 4. *In an optimal direct mechanism, the buyer is recommended to invest if and only if $s < s^*$, even if stochastic (conditional on s) recommendations are permitted.*

The proof proceeds by analyzing the Lagrangian (5). By the Envelope Theorem, (6) implies

$$M'(s) = -\eta_s'(s)\,\mathbb{E}_{|s}[x(s,b)] \leq 0.$$

If $M'(s^*) < 0$, then $M(s) < 0$ for $s > s^*$, and $M(s) > 0$ for $s < s^*$, implying that recommendation $\phi(s) = [s < s^*]$ is the unique maximizer of the Lagrangian.

More subtle is the complementary case, in which $M(s) = 0$ if and only if $s \in I \equiv [\underline{s}, \bar{s}]$ for some $\underline{s} < \bar{s}$, implying $M'(s) = 0$ for $s \in I$, implying $\mathbb{E}_{|s}[x(s,b)] = 0$ for $s \in I$, implying

$$M(s) = -(\delta_b + \lambda)c = 0, \qquad s \in I,$$

implying $\delta_b = 0$ and $\lambda = 0$. If $\delta_b > 0$, an immediate contradiction to $\underline{s} < \bar{s}$ ensues. Suppose that $\delta_b = 0$. In this case, the recommendation ϕ maximizing the Lagrangian is not unique, and one may be concerned that $\lambda = 0$ minimizes the Lagrangian for some maximizer ϕ but not others. It will be shown, however, that if $\lambda = 0$ minimizes the Lagrangian for some maximizer ϕ, then $\lambda = 0$ minimizes the Lagrangian also for a maximizer of the form $\phi = [s < s^*]$. Indeed, set $s^* = \underline{s}$ in $\phi = [s < s^*]$; **OC** is thereby (weakly) relaxed, by insisting that the buyer does not invest for s such that no trade occurs. Thus, with $\phi = [s < \underline{s}]$, **OC** remains nonbinding; that is, $\lambda = 0$ remains optimal.

[31] The dependence of the buyer's virtual valuation η_b on the Lagrange multiplier λ will sometimes be made explicit, for emphasis.

B.4 An Optimal ECN with Multiple Traders

The Model with Multiple Traders

Environment: Specialize the bilateral-trade model described in Section 2 by assuming $\psi = 0$, and extend that model to accommodate multiple buyers and sellers. The exchange mediates trades between S sellers, each owning a unit of the asset, and B latent buyers, each with a unit-demand preference. With some abuse of notation, S and B will denote both sets and their cardinalities. Both S and B are finite. The exchange's payoff is

$$\sum_{i \in S \cup B} t_i + \delta_s \sum_{i \in S} \left(s_i x_i + \psi\left(x_i\right) - t_i \right) + \delta_b \sum_{i \in B} \left(b_i x_i - t_i - c a_i \right),$$

where, for each trader i, t_i is his (possibly negative) payment to the exchange, $x_i \geq 0$ is his allocation of the asset, $a_i \in \{0, 1\}$ is buyer i's investment decision, and $[s_i]_{i \in S}$ and $[b_i]_{i \in B}$ are independently distributed, privately observed types. Feasibility requires $\sum_{i \in S \cup B} x_i \leq S$.

Direct Mechanisms: Applying the Revelation-Principle logic, one can restrict attention to a class of direct mechanisms that generalize the mechanisms of Section 3.1. In particular, first all sellers simultaneously and confidentially report their types to the exchange.[32] Then, the exchange confidentially approaches (or "alerts") buyers one by one, uniformly at random, and asks each approached buyer to invest and to confidentially report his type.[33] Each buyer who has not been approached construes this fact as a recommendation not to invest. Having approached all intended buyers, the exchange enforces trades. Without loss of generality, attention can be restricted to direct mechanisms in which, at equilibrium, each trader is willing to participate in the mechanism before and after having observed his type, each buyer invests if and only if he has been approached, and each trader who observes his type reports it truthfully.

An Optimal ECN with Multiple Traders: The Theorem's Statement

The following theorem describes an optimal ECN, which is a direct mechanism in all but name. In particular, a type report is called an "ask" for a seller and a "bid" for a buyer.

Theorem 7. *An optimal electronic communication network requires partially concealing the order book and restricting the set of sellers' and buyers' orders. It can be implemented by letting each trader choose an order, which specifies an ask or a bid for a unit of the asset. The exchange does not display submitted orders, but instead designs alerts, each of which is a personalized invitations to submit an order. First all sellers simultaneously submit their orders, and then optimally chosen groups of buyers are sequentially alerted to*

[32] The exchange would gain nothing by soliciting sellers' types sequentially. Because no seller invests, no seller requires any feedback that depends on others' reports.

[33] Approaching buyers uniformly at random is motivated by the optimality of pooling their obedience constraints by obscuring from them the order in which they are approached. Here, this feature of an optimal mechanism is assumed without a proof.

invest and submit theirs.[34] *The order-execution rule is designed so that each seller asks, and each buyer bids, his valuation. No subsidies are required.*[35]

To make the theorem's statements precise, additional notation must be introduced. Recall the definitions of virtual valuations η_s and η_b in (4). (In particular, η_b depends on a certain λ, which in the context of the multi-trader model will be interpreted as the Lagrange multiplier on the buyers' pooled obedience constraint.) Recall the augmented cost $\hat{c} \equiv (\delta_b + \lambda) c$. Consider the exchange's hypothetical choice between keeping an asset that it values at some η^* and expending $\hat{c}$ to alert an additional buyer of an unknown type b, to whom the asset can be sold for $\eta_b(b)$. The exchange is indifferent between these two alternatives if η^* is such that

$$\hat{c} = \mathbb{E}_{b|\eta^*}\left[\max\{0, \eta_b(b) - \eta^*\}\right]. \tag{B.6}$$

Define s^* and b^* implicitly by $\eta_s(s^*) = \eta_b(b^*) = \eta^*$.

With this additional notation, the alerting rule in the theorem's statement can be made more precise. In particular, buyers are ordered uniformly at random, and are alerted sequentially until either all B buyers have been alerted or the number of alerted buyers with bids exceeding b^* equals the number of sellers with asks not exceeding s^*.

To describe the order-execution rule, denote seller i's ask by p_i and buyer i's bid by p'_i. Use consecutive positive integers to relabel sellers in the ascending order of their asks. Similarly relabel buyers in the descending orders of their bids. The number of executed trades is denoted by κ and is the largest number i for which $\eta_b(p'_i) > \eta_s(p_i)$, where $p'_i = 0$ if i exceeds the number of alerted buyers, and $p_i = 1$ if i exceeds the number of sellers. Each of the sellers with κ lowest asks sells a unit and is paid $\min\{p_{\kappa+1}, s^*, \eta_s^{-1}(\eta_b(p'_\kappa))\}$. Each of the buyers with κ highest bids buys a unit and pays $\max\left\{p'_{\kappa+1}, \eta_b^{-1}(\eta_s(p_\kappa))\right\}$ if $p'_i < b^*$, and p^* if $p'_i \geq b^*$, for some optimally set p^*. The price p^* is set so as to make the buyer with valuation b^* indifferent between buying the asset for sure at p^* and waiting to compete with the buyers who may submit their orders later.

The exchange can alert buyers in groups, instead of alerting them one by one. Let S' denote the number of sellers with asks less than s^*; that is, $S' = \#\{i \in \{1,..,S\} : p_i < s^*\}$. The exchange alerts the first $\min\{S', B\}$ buyers simultaneously. If $B \leq S'$, no more buyers are alerted. If $B > S'$, define S'' to be the number of sellers who have not sold their items at p^* to the buyers from the first group; that is, $S'' = \#\{i \in \{1,..,S'\} \mid p'_i < b^*\}$. Next, alert $\min\{S'', B - S'\}$ buyers simultaneously. If $B \leq S' + S''$, no more buyers are alerted, and so forth.

[34] The assumption that traders can be alerted asynchronously has counterparts in practice. For example, if Pipeline exchange knows that a trader follows a particular stock, the exchange may send a message to that trader informing him about a trading opportunity in that stock. NASDAQ provides quotes to traders at three levels (Level 1, Level 2, and Level 3); even though these levels do not literally correspond to the alerted groups in the described optimal ECN, they indicate the feasibility of asynchronous alerts.

[35] In contrast to the ECN of Theorem 2, the ECN described here disposes with subsidies by specifying an order-execution rule in which transaction prices differ from asks and bids. In the context of financial markets, the advantage of divorcing asks and bids from transaction prices was first explained by XXXVickrey1961, who introduced the sealed-bid second-price auction.

An Optimal ECN with Multiple Traders: The Theorem's Proof Sketch

The environment resembles search auctions of XXXMcAfee1988, XXXCremer2007, and XXXBoard2010. The main differences are that in the present model, buyers make private investments (hence, information disclosure matters, and "search cost" may arise even if the exchange does not directly care about buyers' investment costs) and multiple items are available for sale, all valued differently by the privately informed sellers. Given the set of investing traders, the environment is as in the model of XXXWilson1985. The proof is only sketched because it resembles the arguments of XXXMcAfee1988 (e.g., the exchanges objective function admits a recursive representation and involves traders' virtual valuations).

The optimal ECN is found by solving a Lagrangian. The Lagrangian maximization problem can be formulated as a costly search problem in which the cost of alerting a buyer is $\hat{c} \equiv (\delta_b + \lambda)\, c$. Here, λ is the Lagrange multiplier on the buyers' pooled obedience constraint in the exchange's problem, analogous to the problem studied in Section 3.2. A single constraint captures the obedience of every buyer because the mechanism treats all buyers symmetrically; each is alerted confidentially and uniformly at random.

To set up the Lagrangian, some additional notation is required. Let $\sigma \equiv [\eta_s(s_i)]_{i=1}^{S}$ denote the ordered vector of all sellers' virtual valuations, where $[s_i]_{i=1}^{S}$ are the sellers' types relabelled using positive integers and ordered in ascending order. Suppose m out of B buyers have not yet been alerted. Let $\beta \equiv [\eta_b(b_i)]_{i=1}^{B}$ denote the ordered vector of all buyers' virtual valuations, where $[b_i]_{i=1}^{B-m}$ are the alerted buyers' valuations relabelled using positive integers and ordered in descending order, and where $b_i = 0$ for each buyer i ($i > B - m$) who has not yet been alerted. Define the exchange's value from executing trades without sending any further alerts by

$$\pi(\sigma, \beta) \equiv \sum_{i=1}^{\min\{S,B\}} \max\{0, \beta_i - \sigma_i\}.$$

This value is weakly increasing in $-\sigma$ and β, where the order on σ and β is the usual product order.

Suppose the exchange chooses how many additional buyers (out of m remaining) to alert, if any. The value of this problem is given by $V_m(\sigma, \beta)$, which can be written recursively by defining, for any vector $(\bar{\sigma}, \bar{\beta})$ of ordered virtual valuations:

$$\begin{aligned} V_n(\bar{\sigma}, \bar{\beta}) &= \max\left\{\pi(\bar{\sigma}, \bar{\beta}), -\hat{c} + \mathbb{E}_{|\bar{\sigma},\bar{\beta}}\left[V_{n-1}(\bar{\sigma}, \bar{\beta} \otimes \tilde{\beta})\right]\right\}, \qquad n = 1, .., m \qquad \text{(B.7)} \\ V_0(\bar{\sigma}, \bar{\beta}) &= \pi(\bar{\sigma}, \bar{\beta}), \end{aligned}$$

where $\tilde{\beta}$ is a random variable (independent from all other random variables) that is equal in distribution to $\eta_b(b_i)$ (for any $i \in B$), and $\bar{\beta} \otimes \tilde{\beta}$ denotes the first B elements of the vector obtained from the vector $(\bar{\beta}, \tilde{\beta})$ by ordering its elements in descending order. The Lagrangian for the exchange's problem is the value of the search problem before a single buyer has been contacted and before sellers have revealed their types: $\mathbb{E}\left[V_B\left(\sigma,\left(\eta_b\left(0\right),...,\eta_b\left(0\right)\right)\right)\right]$.

The derivation of an optimal ECN will use three auxiliary lemmas. In these lemmas, $\tilde{\beta}$ and $\tilde{\beta}'$ are random variables (independent from all other random variables) that are each equal in distribution to $\eta_b\left(b_i\right)$. The following lemma establishes diminishing returns to alerting an additional buyer.

Lemma 5. *Fix σ. Given a vector β and a scalar $\tilde{\beta}'$, define a vector $\beta' \equiv \beta \otimes \tilde{\beta}'$. Then,*

$$\mathbb{E}_{|\sigma,\beta}\left[\pi\left(\sigma,\beta\otimes\tilde{\beta}\right)\right]-\pi\left(\sigma,\beta\right)\geq\mathbb{E}_{|\sigma,\beta'}\left[\pi\left(\sigma,\beta'\otimes\tilde{\beta}\right)\right]-\pi\left(\sigma,\beta'\right).$$

Proof. By

$$\pi\left(\sigma,\beta\otimes\tilde{\beta}\right)=\pi\left(\sigma,\beta\right)+\max\left\{0,\tilde{\beta}-\max\left\{\beta_j,\sigma_j\right\}\right\}$$
$$j=\min\left\{i\in\left\{1,..,\dim\sigma\right\}\mid\beta_i\leq\sigma_i\text{ or }i=\dim\sigma\right\},$$

one can rewrite

$$\mathbb{E}_{|\sigma,\beta}\left[\pi\left(\sigma,\beta\otimes\tilde{\beta}\right)\right]-\pi\left(\sigma,\beta\right)$$

as

$$\mathbb{E}_{|\sigma,\beta}\left[\max\left\{0,\tilde{\beta}-\max\left\{\beta_j,\sigma_j\right\}\right\}\right].$$

Define j' from β' analogously to the way j has been defined from β. Note that $j'\geq j$.
Suppose $j'=j$. Then, by $\beta'_j\geq\beta_j$, $\max\left\{\beta'_j,\sigma_j\right\}\geq\max\left\{\beta_j,\sigma_j\right\}$ implies

$$\mathbb{E}_{|\sigma,\beta}\left[\max\left\{0,\tilde{\beta}-\max\left\{\beta_j,\sigma_j\right\}\right\}\right]\geq\mathbb{E}_{|\sigma,\beta'}\left[\max\left\{0,\tilde{\beta}-\max\left\{\beta'_j,\sigma_j\right\}\right\}\right].$$

Suppose $j'>j$. Then, $j'=j+1$, $\beta_j\leq\sigma_j$ and $\beta'_{j+1}=\beta_j$, which, together with $\sigma_{j+1}\geq\sigma_j$, imply

$$\max\left\{\beta_j,\sigma_j\right\}=\sigma_j\leq\max\left\{\beta'_{j'},\sigma_{j'}\right\}=\max\left\{\beta_j,\sigma_{j+1}\right\},$$

so that, again,

$$\mathbb{E}_{|\sigma,\beta}\left[\max\left\{0,\tilde{\beta}-\max\left\{\beta_j,\sigma_j\right\}\right\}\right]\geq\mathbb{E}_{|\sigma,\beta'}\left[\max\left\{0,\tilde{\beta}-\max\left\{\beta'_j,\sigma_j\right\}\right\}\right].$$

□

The following lemma establishes the **one-period look-ahead property**: The exchange optimally alerts an additional buyer if and only if doing so would be optimal even if this additional buyer were the last one remaining.

Lemma 6. *For any σ and β, the exchange optimally alerts another available buyer if and only if*

$$\hat{c}<\mathbb{E}_{|\sigma,\beta}\left[\pi\left(\sigma,\beta\otimes\tilde{\beta}\right)\right]-\pi\left(\sigma,\beta\right),$$

*which defines the **myopic rule**.*

Proof. The proof is by induction.

For the inductive base, let $m = 1$. The exchange alerts the remaining buyer if and only if

$$\hat{c} < \mathbb{E}_{|\sigma,\beta}\left[\pi\left(\sigma, \beta \otimes \tilde{\beta}\right)\right] - \pi\left(\sigma, \beta\right),$$

as desired.

For the inductive step, suppose the myopic rule is optimal if m or fewer buyers remain. It will be shown that it is optimal also if $m+1$ buyers remain.

If $m+1$ buyers remain and $\hat{c} < \mathbb{E}_{|\sigma,\beta}\left[\pi\left(\sigma, \beta \otimes \tilde{\beta}\right)\right] - \pi\left(\sigma, \beta\right)$, the exchange's payoff from alerting another buyer is at least his payoff from alerting exactly one more buyer:

$$-\hat{c} + \mathbb{E}_{|\sigma,\beta}\left[\pi\left(\sigma, \beta \otimes \tilde{\beta}\right)\right] > \pi\left(\sigma, \beta\right),$$

where the inequality is by hypothesis. Hence, the exchange optimally alerts at least one more buyer.

If $m+1$ buyers remain and $\hat{c} > \mathbb{E}_{|\sigma,\beta}\left[\pi\left(\sigma, \beta \otimes \tilde{\beta}\right)\right] - \pi\left(\sigma, \beta\right)$, the exchange's payoff from alerting another buyer is

$$-\hat{c} + \mathbb{E}_{|\sigma,\beta}\left[V_m\left(\sigma, \beta \otimes \tilde{\beta}\right)\right] = -\hat{c} + \mathbb{E}_{|\sigma,\beta}\left[\pi\left(\sigma, \beta \otimes \tilde{\beta}\right)\right] < \pi\left(\sigma, \beta\right),$$

where the equality follows from Lemma 5 and from the hypothesis that the myopic rule is optimal at m, and where the inequality is by hypothesis. Hence, the exchange does not alert additional buyers.

The inductive argument thus establishes that the myopic rule is optimal for every m. □

The final lemma, a monotonicity result, will be used to show that it is optimal for traders to bid and ask their valuations.

Lemma 7. *A buyer's expected trade is weakly increasing in his valuation. A seller's expected trade is weakly decreasing in his valuation.*

Proof. A stronger result will be proved. Fix the type profile that determines the realization of the traders' valuations. The stated monotonicity of expected trades shall be established conditional on any type profile.

By contradiction, suppose seller i trades if he reports s_i^A (Scenario A), in which case the exchange's expected payoff is Π^A, and seller i does not trade if he reports s_i^B with $s_i^B < s_i^A$ (Scenario B), in which case the exchange's expected payoff is Π^B. On the one hand, $\Pi^B \leq \Pi^A$ because Π^B is attainable in Scenario A by adhering to the same alerting rule as in Scenario B and refusing to trade with seller i. On the other hand, $\Pi^B > \Pi^A$ because, in Scenario B, the exchange can adopt the same alerting and allocation rules as in Scenario A, thereby enjoying a higher payoff when trading with player i (by $s_i^B < s_i^A$). A contradiction is reached, implying that a seller with a lower valuation is more likely to trade.

An analogous "revealed-preference" argument implies that a buyer with a higher valuation is more likely to trade. □

One can now derive the features of the optimal ECN described in Theorem 7.

Alerting Buyers: By Lemma 6, the exchange optimally alerts a latent buyer if and only if

$$\hat{c} < \mathbb{E}_{|\sigma,\beta}\left[\pi\left(\sigma,\beta\otimes\tilde{\beta}\right)\right] - \pi\left(\sigma,\beta\right),$$

or equivalently,

$$\hat{c} < \begin{cases} \mathbb{E}_{|\sigma,\beta}\left[\max\left\{0,\tilde{\beta} - \min\left\{\sigma_i : \sigma_i > \beta_i\right\}\right\}\right] & \text{if } \beta_S < \sigma_S \\ \mathbb{E}_{|\sigma,\beta}\left[\max\left\{0,\tilde{\beta} - \beta_S\right\}\right] & \text{if } \beta_S > \sigma_S, \end{cases}$$

which—using the definition of η^* in (B.6)—can be rewritten as:

$$\beta_S < \sigma_S \text{ and } \min\left\{\sigma_i : \sigma_i > \beta_i\right\} < \eta^* \qquad \text{or} \qquad \beta_S > \sigma_S \text{ and } \beta_S < \eta^*.$$

In other words, a buyer is recommended to invest as long as the virtual valuation of the displaced trader (buyer or seller) does not exceed η^*. Thus, sellers with virtual valuations exceeding η^* never trade, and buyers are recommended to invest until $|\{i : \sigma_i < \eta^*\}|$ buyers with virtual valuations exceeding η^* have been identified or until all B buyers have been alerted (whichever occurs first).

Payments: The allocation and alert rules follow from the pointwise maximization of the Lagrangian. The payments are constructed by the standard envelope argument.

Equilibrium Strategies: Each seller participates because the payoff of any seller's type is nonnegative. An alerted buyer invests; his obedience constraint is satisfied because the Lagrangian is maximized at the saddle-point value of the Lagrange multiplier. Alerted buyer i pays

$$[\beta_i > \sigma_i]\,\eta_b^{-1}\left(\max\left\{\beta_{\kappa+1},\sigma_\kappa\right\}\right).$$

Hence, his payoff is

$$[\beta_i > \sigma_i]\left(\eta_b^{-1}\left(\beta_i\right) - \eta_b^{-1}\left(\max\left\{\beta_{\kappa+1},\sigma_\kappa\right\}\right)\right),$$

which is nonnegative because $\beta_i \geq \max\left\{\beta_{\kappa+1},\sigma_\kappa\right\}$.

Suppose buyer i buys if he reports truthfully. By Lemma 7, he still buys if he overstates his valuation, and his payment is unchanged. Therefore, he can affect his allocation and payment only by understating sufficiently his valuation, in which case he does not buy and pays nothing, finding himself weakly worse off.

Suppose buyer i does not buy if he reports truthfully. By Lemma 7, he affects his allocation and payment only by overstating sufficiently his valuation, in which case his payoff is at most $\eta_b^{-1}\left(\beta_i\right) - \eta_b^{-1}\left(\beta_\kappa\right)$, which is nonpositive because $\beta_i \leq \beta_\kappa$.

Seller i receives $[\beta_i > \sigma_i]\, \eta_s^{-1}(\min\{\sigma_{\kappa+1}, \beta_\kappa\})$. Hence, his payoff is

$$[\beta_i > \sigma_i]\left(\eta_s^{-1}(\min\{\sigma_{\kappa+1}, \beta_\kappa\}) - \eta_s^{-1}(\sigma_i)\right),$$

which is nonnegative because $\sigma_i \le \min\{\beta_\kappa, \sigma_{\kappa+1}\}$.

Suppose seller i sells if he reports truthfully. By Lemma 7, he still sells if he understates his valuation, and his payment is unchanged. Thus, he can affect his allocation and payment only by overstating sufficiently his valuation, in which case he does not sell and pays nothing, finding himself weakly worse off.

Suppose seller i does not sell when he reports truthfully. By Lemma 7, he affects his allocation and payment only by understating sufficiently his valuation, in which case his payoff is at most $\eta_s^{-1}(\sigma_\kappa) - \eta_s^{-1}(\sigma_i)$, which is nonpositive because $\sigma_i \ge \sigma_\kappa$.

Proof of Theorem 4

Consider an ECN that has an open order book and is constructed to maximize the total surplus while disregarding both the obedience constraint (which will hold) and the zero-revenue constraint (which will hold approximately). In that case, $\eta_s(s) = s$ and $\eta_b(b) = b$ for all s and b.

Let $X_i(b)$ denote the conditional probability that buyer i with valuation b obtains the item. The probability is conditional on all orders that are submitted before buyer i is alerted. By the Envelope Theorem, the buyer's expected payoff from trade is

$$\begin{aligned}\int_0^1 X_i(b)(1 - F_b(b))\,\mathrm{d}b &= \int_0^{b^*} X_i(b)(1 - F_b(b))\,\mathrm{d}b + \int_{b^*}^1 (1 - F_b(b))\,\mathrm{d}b \\ &= \int_0^{b^*} X_i(b)(1 - F_b(b))\,\mathrm{d}b + c,\end{aligned}$$

where the first equality uses the fact that $X_i(b) = 1$ for $b \ge b^*$, and the second equality is by (B.6) with $\lambda = 0$ and $\delta_b = 1$. Because the expression displayed above weakly exceeds c, the buyer invests when alerted. (He does not invest when not alerted because in that case, $X_i(b) = 0$ for any b.) Thus, the obedience constraint holds.[36]

It remains to verify truthfulness. Consider seller i with valuation s who asks p. His payoff is

$$[\min\{s_{\kappa+1}, b_\kappa, s^*\} - p > 0]\,(\min\{s_{\kappa+1}, b_\kappa, s^*\} - s),$$

which makes it optimal to ask $p = s$ even if the seller knows the realizations of others' valuations.

[36]Proposition 1 of XXXAthey2007 and the discussion that precedes it establish that, under some conditions, in a dynamic extension of a VCG mechanism (called the "team mechanism") players have incentives to take efficient actions. Here, these conditions do not apply; the mechanism is not a team mechanism because a buyer's investment is more profitable for him than it is for the exchange; the buyer does not internalize other buyers' investment costs. Hence, a direct proof is required.

Consider buyer i with valuation b who bids p'. If $p' \geq b^*$, the ECN sells to him at the price p^* satisfying

$$b^* - p^* = Q\mathbb{E}\left[b^* - \max\left\{b_{\kappa+1}, s_\kappa\right\} \mid \text{auction occurs}\right],$$

where Q is the probability that the auction occurs; that is, not all investing buyers buy at price p^*. Here, both the probability and the expectation are conditional on the orders that have been submitted before buyer i was alerted. For $b < b^*$, the definition of p^* implies

$$\begin{aligned}
b - p^* &\leq Q\mathbb{E}_{|b}\left[b - \max\left\{b_{\kappa+1}, s_\kappa\right\} \mid \text{auction occurs}\right] \\
&\leq Q\mathbb{E}_{|b}\left[\max\left\{0, b - \max\left\{b_{\kappa+1}, s_\kappa\right\}\right\} \mid \text{auction occurs}\right] \\
&= Q\mathbb{E}_{|b}\left[\max_{p' \leq b^*}\left\{\left[p' - \max\left\{b_{\kappa+1}, s_\kappa\right\} \geq 0\right]\left(b - \max\left\{b_{\kappa+1}, s_\kappa\right\}\right)\right\} \mid \text{auction occurs}\right].
\end{aligned}$$

Thus, buyer i with $b < b^*$ does not gain from bidding $p' \geq b^*$ and buying immediately at p^*.

Consider buyer i with $b > b^*$. The definition of p^* implies

$$\begin{aligned}
b - p^* &\geq Q\mathbb{E}_{|b}\left[b_i - \max\left\{b_{\kappa+1}, s_\kappa\right\} \mid \text{auction occurs}\right] \\
&= Q\mathbb{E}_{|b}\left[\max\left\{0, b - \max\left\{b_{\kappa+1}, s_\kappa\right\}\right\} \mid \text{auction occurs}\right] \\
&= Q\mathbb{E}_{|b}\left[\max_{p' \leq b^*}\left\{\left[p' - \max\left\{b_{\kappa+1}, s_\kappa\right\} \geq 0\right]\left(b_i - \max\left\{b_{\kappa+1}, s_\kappa\right\}\right)\right\} \mid \text{auction occurs}\right].
\end{aligned}$$

Thus, buyer i does not gain from bidding $p' \neq b$; the ECN is truthful.

Even though in general exchanges, the Myerson-Satterthwaite Theorem (XXXMyerson1983a) implies that budget balance cannot be guaranteed, under some conditions, approximate budget balance obtains as the number of traders grows—as has been shown by XXXGresik1989. To apply the ideas of XXXGresik1989 to the present model, it will be argued that, as $\tau \to \infty$, transaction prices converge to b^* when $B_0 \geq S_0 F_s(b^*) / (1 - F_b(b^*))$, and to a certain price $\hat{p}$ otherwise, where $\hat{p}$ solves

$$S_0 F_s(\hat{p}) = B\left(1 - F_b(\hat{p})\right). \tag{B.8}$$

Consider the case with $B_0 \geq S_0 F_s(b^*) / (1 - F_b(b^*))$. As $\tau \to \infty$, the Law of Large Numbers applies (see the proof of Theorem 4 of XXXGresik1989), and approximately $SF_s(b^*)$ sellers are available (by $s^* = b^*$), each of whom causes the ECN to alert approximately $1/(1 - F_b(b^*))$ buyers. Thus, the total number of alerted buyers is approximately $SF_s(b^*) / (1 - F_b(b^*))$; they transact at price b^*; the approximate budget deficit is zero.

Consider the case with $B_0 < S_0 F_s(b^*) / (1 - F_b(b^*))$. Because more sellers than buyers are willing to trade at price b^*, the buyer with valuation b^* who understates his valuation will surely participate in an auction, and will win. Therefore, p^*—the price at which a buyer can trade immediately—must equal the price at which that buyer would win the auction, which is $\hat{p}$ in (B.8). Thus, all transactions are at the price $\hat{p}$. The approximate budget deficit is zero.

B.5 An Optimal Mechanism Subject to an Open Order Book

For the bilateral-trade environment of Section 2 with $\psi \equiv 0$, Theorem 8 shall derive an optimal ECN subject to an open order book. Such an ECN may emerge as an outcome of the regulator's insistence on an open order book or as an outcome of unravelling due to exchange competition with limited commitment, studied in Section 4.1.

The theorem shall use threshold parameters s_1^* and s_2^* defined implicitly by

$$\int_{n_b^{-1}(\eta_s(s_1^*))}^{1} (1 - F_b(b))\, \mathrm{d}b = c \qquad \text{and} \qquad n_b(\eta_s(s_2^*)) = \eta_s(s_1^*), \qquad \text{where} \tag{B.9}$$

$$n_b(b) \equiv \eta_b(b, 0) = b - (1 - \delta_b) \frac{1 - F_b(b)}{f_b(b)},$$

implying $0 < s_1^* \leq s_2^* < 1$, with $s_1^* < s_2^*$ if and only if $\delta_b < 1$.

Theorem 8. *In an electronic communication network that is optimal subject to an open order book, the seller may submit a limit sell order for one unit of the asset by specifying an ask p. If and only if $p < s_2^*$, the buyer is recommended to invest. Then the buyer decides whether to invest and submit a limit buy order for one unit of the asset by specifying a bid, p'. If and only if $p' \geq n_b^{-1}(\eta_s(\min\{s_1^*, p\}))$, trade occurs, the buyer pays $n_b^{-1}(\eta_s(\min\{s_1^*, p\}))$, and the seller receives amount*

$$s_2^* + \left[\max\left\{p, \eta_s^{-1}(n_b(p'))\right\} \leq s_1^*\right] \left(\eta_s^{-1}(n_b(p')) - s_2^*\right).$$

In equilibrium, the buyer invests if and only if he is recommended to do so. The seller asks, and the buyer bids, his valuation.

Proof. The proof is in two steps. Step 1 computes the optimal allocation rule. Step 2 shows that this allocation rule is implemented by the payments described in the theorem.

Step 1

The exchange maximizes:

$$\int_0^1 \phi(s) \left(\int_0^1 x(s,b)(n_b(b) - \eta_s(s)) f_b(b)\, \mathrm{d}b - \delta_b c \right) f_s(s)\, \mathrm{d}s$$

subject to the buyer's obedience constraint with full disclosure of the seller's type:

$$\phi(s) f_s(s) \left(\int_0^1 x(b,s)(1 - F_b(b))\, \mathrm{d}b - c \right) \geq 0, \qquad \text{for all } s, \tag{B.10}$$

and subject to appropriate monotonicity constraints. Guess that the solution is given by

$$x(b,s) = [n_b(b) > \eta_s(\min\{s, s_1^*\}) \text{ and } s < s_2^*] \tag{B.11}$$

$$\phi(s) = [s < s_2^*], \tag{B.12}$$

where s_1^* and s_2^* are defined in (B.9). The implied seller's and buyer's expected allocations are monotone in their valuations, as required by incentive-compatibility. It remains to show that (B.11) and (B.12) solve the exchange's relaxed problem, which neglects the monotonicity constraints.

For each s, let $\lambda(s)$ be the Lagrange multiplier on (B.10). The Lagrangian is:

$$\int_0^1 \phi(s) \left(\int_0^1 x(s,b) \left(\eta_b(b, \lambda(s)) - \eta_s(s)\right) f_b(b) \, \mathrm{d}b - (\delta_b + \lambda(s)) c \right) f_s(s) \, \mathrm{d}s. \tag{B.13}$$

Conjecture that

$$\lambda(s) = \left\langle (\eta_s(s) - \eta_s(s_1^*)) \frac{f_b\left(n_b^{-1}(\eta_s(s_1^*))\right)}{1 - F_b\left(n_b^{-1}(\eta_s(s_1^*))\right)} \right\rangle, \tag{B.14}$$

where the bracket is defined by $\langle y \rangle \equiv \max\{0, \min\{1-\delta_b, y\}\}$ for any y.

First, it will be shown that, given the Lagrange multipliers conjectured in (B.14), the allocation rule conjectured in (B.11) and (B.12) maximizes the Lagrangian in (B.13). Suppose that $s \leq s_1^*$. Then, $\lambda(s) = 0$, and $x(b,s) = [n_b(b) > \eta_s(s)]$, as conjectured in (B.11). The term multiplying $\phi(s)$ in (B.12) satisfies

$$\int_0^1 \max\{0, n_b(b) - \eta_s(s)\} f_b(b) \, \mathrm{d}b - \delta_b c \geq 0,$$

implying $\phi(s) = 1$, as conjectured in (B.12).

Suppose $s \in (s_1^*, s_2^*)$. Then, $\lambda(s) \in (0, 1-\delta_b)$. If $n_b(b) > \eta_s(s_1^*)$, then

$$\eta_b(b, \lambda(s)) - \eta_s(s) > \eta_b\left(n_b^{-1}(\eta_s(s_1^*)), \lambda(s)\right) - \eta_s(s) = 0,$$

where the inequality is a consequence of $n_b(b) > \eta_s(s_1^*)$ and $1 - \delta_b - \lambda(s) > 0$, and the equality is by the construction of $\lambda(s)$. Hence, $x(b,s) = 1$, as conjectured in (B.11). If $n_b(b) \leq \eta_s(s_1^*)$, then

$$\eta_b(b, \lambda(s)) - \eta_s(s) \leq \eta_b\left(n_b^{-1}(\eta_s(s_1^*)), \lambda(s)\right) - \eta_s(s) = 0.$$

Hence, $x(b,s) = 0$, as conjectured in (B.11). Finally, the term multiplying $\phi(s)$ in (B.13) satisfies

$$\begin{aligned} &\int_{n_b^{-1}(\eta_s(s_1^*))}^1 \left(\eta_b(b, \lambda(s)) - \eta_s(s)\right) f_b(b) \, \mathrm{d}b - (\delta_b + \lambda(s)) c \\ &= \left(n_b^{-1}(\eta_s(s_1^*)) - \eta_s(s)\right) \left(1 - F_b\left(n_b^{-1}(\eta_s(s_1^*))\right)\right) > 0, \end{aligned}$$

by the definitions of s_1^* and s_2^*. Thus, $\phi(s) = 1$, as conjectured in (B.12).

Suppose $s \geq s_2^*$. Then, $\lambda(s) = 1 - \delta_b$. The term multiplying $\phi(s)$ in (B.13) equals:

$$\int_{\eta(s)}^1 (b - \eta_s(s)) f_b(b) \, \mathrm{d}b - c = \int_{\eta(s)}^1 (1 - F_b(b)) \, \mathrm{d}b - c \leq 0,$$

which makes $\phi(s) = 0$ optimal, as conjectured in (B.12).

It remains to show that given the allocation rule conjectured in (B.11) and (B.12), the Lagrange multipliers conjectured in (B.14) minimize the Lagrangian in (B.13). When $s \leq s_1^*$, the coefficient multiplying $\lambda(s)$ in the Lagrangian (B.13) is

$$\phi(s)\left(\int_{n_b^{-1}(\eta_s(s))}^{1} (1 - F_b(b))\,\mathrm{d}b - c\right) \geq 0,$$

so setting $\lambda(s) = 0$ is optimal, as conjectured in (B.14). When $s \geq s_1^*$, the corresponding coefficient is

$$\phi(s)\left(\int_0^1 x(b,s)(1 - F_b(b))\,\mathrm{d}b - c\right) = 0$$

(either because the obedience constraint binds or because $\phi(s) = 0$), so any $\lambda(s)$ is optimal, including the one conjectured in (B.14).

Because the conjectured allocation rule and the Lagrange multipliers constitute a saddle point of the Lagrangian, the conjectured allocation solves the problem for which the Lagrangian has been set up.

Step 2

It remains to show that the payments described in the theorem induce each trader to bid or ask his valuation.

Suppose $s \geq s_2^*$. If the seller reports $p \geq s_2^*$, his expected payoff is 0. If the seller reports $p \in (s_1^*, s_2^*)$, his expected payoff is

$$\left(1 - F_b\left(n_b^{-1}\left(\eta_s\left(s_1^*\right)\right)\right)\right)\left(s_2^* - s\right) < 0.$$

If the seller reports $p < s_1^*$, his expected payoff is

$$\begin{aligned}\left(1 - F_b\left(n_b^{-1}\left(\eta_s\left(s_1^*\right)\right)\right)\right)\left(s_2^* - s\right) + \int_{n_b^{-1}(\eta_s(p))}^{n_b^{-1}(\eta_s(s_1^*))}\left(\eta_s^{-1}\left(n_b(b)\right) - s\right) f_b(b)\,\mathrm{d}b \\ \leq \int_{n_b^{-1}(\eta_s(p))}^{n_b^{-1}(\eta_s(s_1^*))}\left(\eta_s^{-1}\left(n_b(b)\right) - s_2^*\right) f_b(b)\,\mathrm{d}b < 0\end{aligned}$$

because $\eta_s^{-1}\left(n_b\left(n_b^{-1}\left(\eta_s\left(s_1^*\right)\right)\right)\right) = s_1^* \leq s_2^*$. Thus, the seller does not gain from lying when $s \geq s_2^*$.

Suppose $s \in (s_1^*, s_2^*)$. If the seller reports $p \in (s_1^*, s_2^*)$, his expected payoff is

$$\left(1 - F_b\left(n_b^{-1}\left(\eta_s\left(s_1^*\right)\right)\right)\right)\left(s_2^* - s\right) \geq 0,$$

so reporting $p = s$ is optimal on (s_1^*, s_2^*). If the seller reports $p \geq s_2^*$, his expected payoff is 0. If the seller reports $p \leq s_1^*$, his gain from lying is

$$\begin{aligned}
&\left(1 - F_b\left(n_b^{-1}\left(\eta_s\left(s_1^*\right)\right)\right)\right)\left(s_2^* - s\right) + \int_{n_b^{-1}(\eta_s(p))}^{n_b^{-1}(\eta_s(s_1^*))} \left(\eta_s^{-1}\left(n_b\left(b\right)\right) - s\right) f_b\left(b\right) \mathrm{d}b \\
&\qquad\qquad\qquad\qquad - \left(1 - F_b\left(n_b^{-1}\left(\eta_s\left(s_1^*\right)\right)\right)\right)\left(s_2^* - s\right) \\
&= \int_{n_b^{-1}(\eta_s(p))}^{n_b^{-1}(\eta_s(s_1^*))} \left(\eta_s^{-1}\left(n_b\left(b\right)\right) - s\right) f_b\left(b\right) \mathrm{d}b \leq \int_{n_b^{-1}(\eta_s(p))}^{n_b^{-1}(\eta_s(s_1^*))} \left(\eta_s^{-1}\left(n_b\left(b\right)\right) - s_1^*\right) f_b\left(b\right) \mathrm{d}b \leq 0
\end{aligned}$$

because $\eta_s^{-1}\left(n_b\left(n_b^{-1}\left(\eta_s\left(s_1^*\right)\right)\right)\right) = s_1^*$. Thus, the seller does not gain from lying when $s \in (s_1^*, s_2^*)$.

Suppose $s \leq s_1^*$. If the seller reports $p \leq s_1^*$, his payoff is

$$\left(1 - F_b\left(n_b^{-1}\left(\eta_s\left(s_1^*\right)\right)\right)\right)\left(s_2^* - s\right) + \int_{n_b^{-1}(\eta_s(p))}^{n_b^{-1}(\eta_s(s_1^*))} \left(\eta_s^{-1}\left(n_b\left(b\right)\right) - s\right) f_b\left(b\right) \mathrm{d}b \geq 0,$$

where the first term is independent of p, and the second term is maximized at $p = s$. If the seller reports $p \in (s_1^*, s_2^*)$, his gain relative to being truthful is

$$\begin{aligned}
&\left(1 - F_b\left(n_b^{-1}\left(\eta_s\left(s_1^*\right)\right)\right)\right)\left(s_2^* - s\right) \\
&\quad - \left[\left(1 - F_b\left(n_b^{-1}\left(\eta_s\left(s_1^*\right)\right)\right)\right)\left(s_2^* - s\right) + \int_{n_b^{-1}(\eta_s(s))}^{n_b^{-1}(\eta_s(s_1^*))} \left(\eta_s^{-1}\left(n_b\left(b\right)\right) - s\right) f_b\left(b\right) \mathrm{d}b\right] \\
&\qquad\qquad\qquad\qquad = -\int_{n_b^{-1}(\eta_s(s))}^{n_b^{-1}(\eta_s(s_1^*))} \left(\eta_s^{-1}\left(n_b\left(b\right)\right) - s\right) f_b\left(b\right) \mathrm{d}b < 0.
\end{aligned}$$

If the seller reports $p \geq s_2^*$, no trade occurs and his payoff is zero. Thus, the seller does not gain from lying when $s \leq s_1^*$.

The payoff of the buyer whose value is b and who reports p' is

$$\left[p' \geq n_b^{-1}\left(\eta_s\left(\min\left\{s_1^*, p\right\}\right)\right)\right]\left(b - n_b^{-1}\left(\eta_s\left(\min\left\{s_1^*, p\right\}\right)\right)\right),$$

which is maximized by setting $p' = b$. □

In Theorem 8, when $s \leq s_1^*$, the buyer's obedience constraint does not bind, and the ECN sells the asset if and only if the buyer's virtual valuation exceeds the seller's virtual valuation. When $s \in (s_1^*, s_2^*)$, the obedience constraint binds and the ECN caps the buyer's price at $n_b^{-1}\left(\eta_s\left(s_1^*\right)\right)$ to induce him to invest. When $s \geq s_2^*$, the gain generated from trade is insufficient to compensate the buyer for his investment, so he does not invest. As for the seller's payment, there is much freedom in specifying it; only its expected value given his order matters. In Theorem 8, the seller receives amount s_2^*, unless both traders submit orders at sufficiently low prices, in which case he receives less.

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